EXAMPLE 4.7

The open loop transfer function of a unity feedback system is given by G(s) = 1/s(1+s) (1+2s). Sketch the polar plot and determine the gain margin and phase margin.

SOLUTION

Given that, G(s) = 1/s(1+s) (1 +2s)

Put s = jω.

$$\therefore G(j\omega) = \frac{1}{j\omega\,(1+j\omega)\,(1+j2\omega)}$$

The corner frequencies are ω_{c1} = 1/2 = 0.5 rad/sec and ω_{c2} = 1 rad/sec. The magnitude and phase angle of G(jω) are calculated for the corner frequencies and for frequencies around corner frequencies and tabulated in table-1. Using polar to rectangular conversion, the polar coordinates listed in table-1 are converted to rectangular coordinates and tabulated in table-2. The polar plot using polar coordinates is sketched on a polar graph sheet as shown in fig 4.7.1. The polar plot using rectangular coordinates is sketched on an ordinary graph sheet as shown in fig 4.7.2.

$$G(j\omega) = \frac{1}{(j\omega)\,(1+j\omega)\,(1+j2\omega)} = \frac{1}{\omega\angle 90^\circ\ \sqrt{1+\omega^2}\ \angle\tan^{-1}\omega\ \sqrt{1+4\omega^2}\ \angle\tan^{-1}2\omega}$$

$$= \frac{1}{\omega\sqrt{(1+\omega^2)\,(1+4\omega^2)}}\ \angle -90^\circ - \tan^{-1}\omega - \tan^{-1}2\omega$$

$$\therefore |G(j\omega)| = \frac{1}{\omega\sqrt{(1+\omega^2)\,(1+4\omega^2)}} = \frac{1}{\omega\sqrt{1+4\omega^2+\omega^2+4\omega^4}} = \frac{1}{\omega\sqrt{1+5\omega^2+4\omega^4}}$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1}\omega - \tan^{-1}2\omega$$

TABLE-1 : Magnitude and phase of G(jω) at various frequencies

ω rad/sec	0.35	0.4	0.45	0.5	0.6	0.7	1.0
|G(jω)|	2.2	1.8	1.5	1.2	0.9	0.7	0.3
∠G(jω) deg	−144	−150	−156	−162	−171	−179.5 ≈ −180	−198

TABLE-2 : Real and imaginary part of G(jω) at various frequencies

ω rad/sec	0.35	0.4	0.45	0.5	0.6	0.7	1.0
$G_R(j\omega)$	−1.78	−1.56	−1.37	−1.14	−0.89	−0.7	−0.29
$G_I(j\omega)$	−1.29	−0.9	−0.61	−0.37	−0.14	0	0.09

RESULT

Gain margin, K_g = 1.4286

Phase margin, γ = +12°

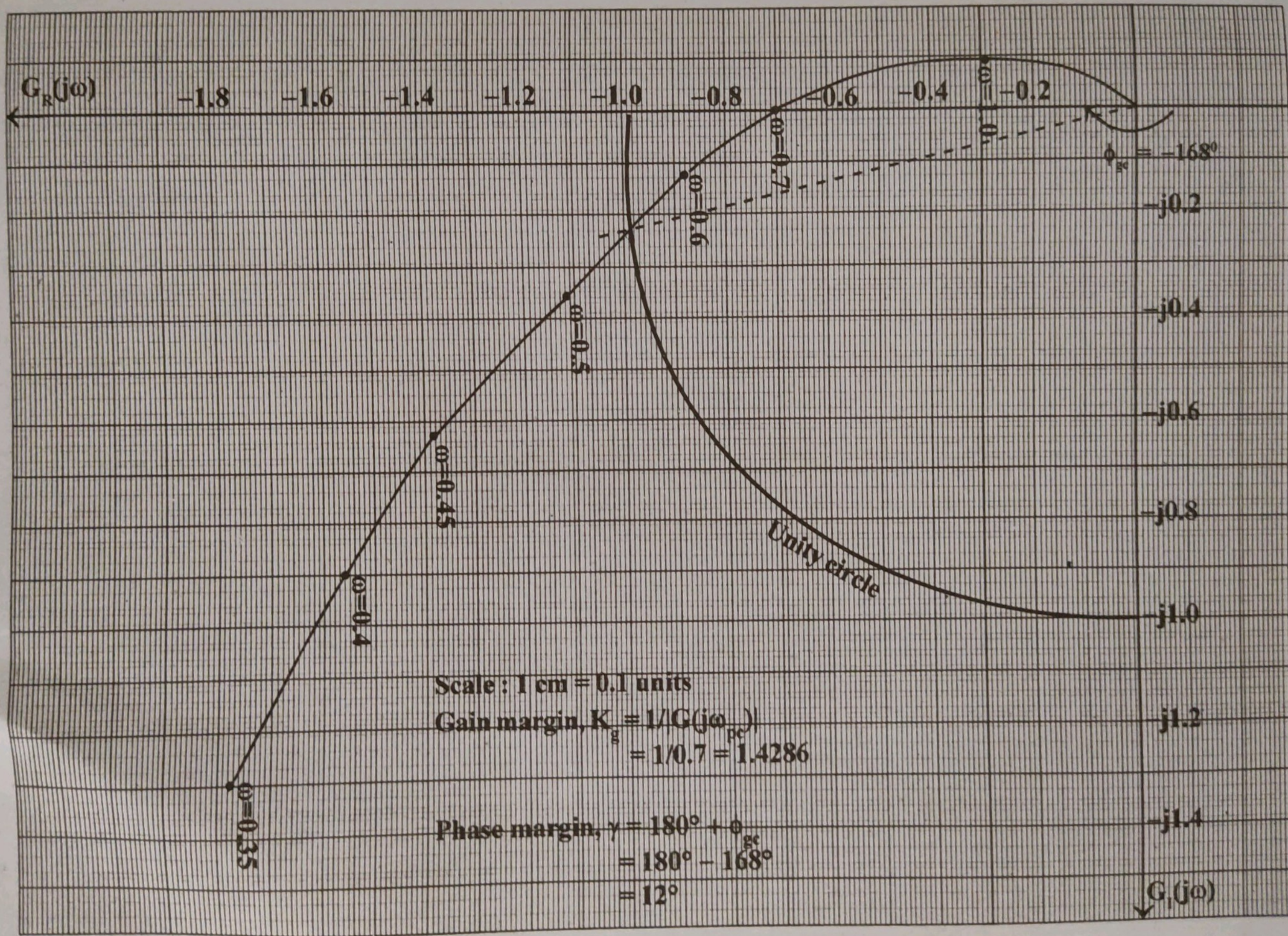


Fig 4.7.2: Polar plot of, $G(j\omega) = \dfrac{1}{j\omega\,(1+j\omega)\,(1+j2\omega)}$ *(using rectangular coordinates).*

EXAMPLE 4.8

The open loop transfer function of a unity feedback system is given by $G(s) = 1/s^2(1+s)(1+2s)$. Sketch the polar plot and determine the gain margin and phase margin.

SOLUTION

Given that, $G(s) = 1/s^2(1+s)(1+2s)$

Put $s = j\omega$, $\quad \therefore G(j\omega) = \dfrac{1}{(j\omega)^2 (1+j\omega)(1+j2\omega)}$

The corner frequencies are $\omega_{c1} = 0.5$ rad/sec and $\omega_{c2} = 1$ rad/sec. The magnitude and phase angle of $G(j\omega)$ are calculated for the corner frequencies and frequencies around corner frequencies are tabulated in table-1. Using the polar to rectangular conversion, the polar coordinates listed in table-1 are converted to rectangular coordinates and tabulated in table-2. The polar plot using polar coordinates is sketched on a polar graph sheet as shown in fig 4.8.1. The polar plot using rectangular coordinates is sketched on an ordinary graph sheet as shown in fig 4.8.2.

$$G(j\omega) = \frac{1}{(j\omega)^2 (1+j\omega)(1+j2\omega)}$$

$$= \frac{1}{\omega^2\angle 180° \sqrt{1+\omega^2}\ \angle\tan^{-1}\omega\ \sqrt{1+4\omega^2}\ \angle\tan^{-1}2\omega}$$

$$G(j\omega) = \frac{1}{\omega^2\sqrt{1+\omega^2}\ \sqrt{1+4\omega^2}}\ \angle(-180 - \tan^{-1}\omega - \tan^{-1}2\omega)$$

$$|G(j\omega)| = \frac{1}{\omega^2\sqrt{1+\omega^2}\ \sqrt{1+4\omega^2}} = \frac{1}{\omega^2\sqrt{(1+\omega^2)(1+4\omega^2)}}$$

$$= \frac{1}{\omega^2\sqrt{1+5\omega^2+4\omega^4}}$$

$$\angle G(j\omega) = -180° - \tan^{-1}\omega - \tan^{-1}2\omega.$$

TABLE-1 : Magnitude and phase plot of G(jω) at various frequencies

ω rad/sec	0.45	0.5	0.55	0.6	0.65	0.7	0.75	1.0
|G(jω)|	3.3	2.5	1.9	1.5	1.2	0.97 ≈ 1	0.8	0.3
∠G(jω) deg	–246	–251	–256	–261	–265	–269	–273	–288

TABLE-2 : Real and imaginary part of G(jω)

ω rad/sec	0.45	0.5	0.55	0.6	0.65	0.7	0.75	1.0
$G_R(j\omega)$	–1.34	–0.81	–0.46	–0.23	–0.1	–0.02	0.04	0.09
$G_I(j\omega)$	3.01	2.36	1.84	1.48	1.2	1.0	0.8	0.29

RESULT

Gain margin, $K_g = 0$

Phase margin, $\gamma = -90°$

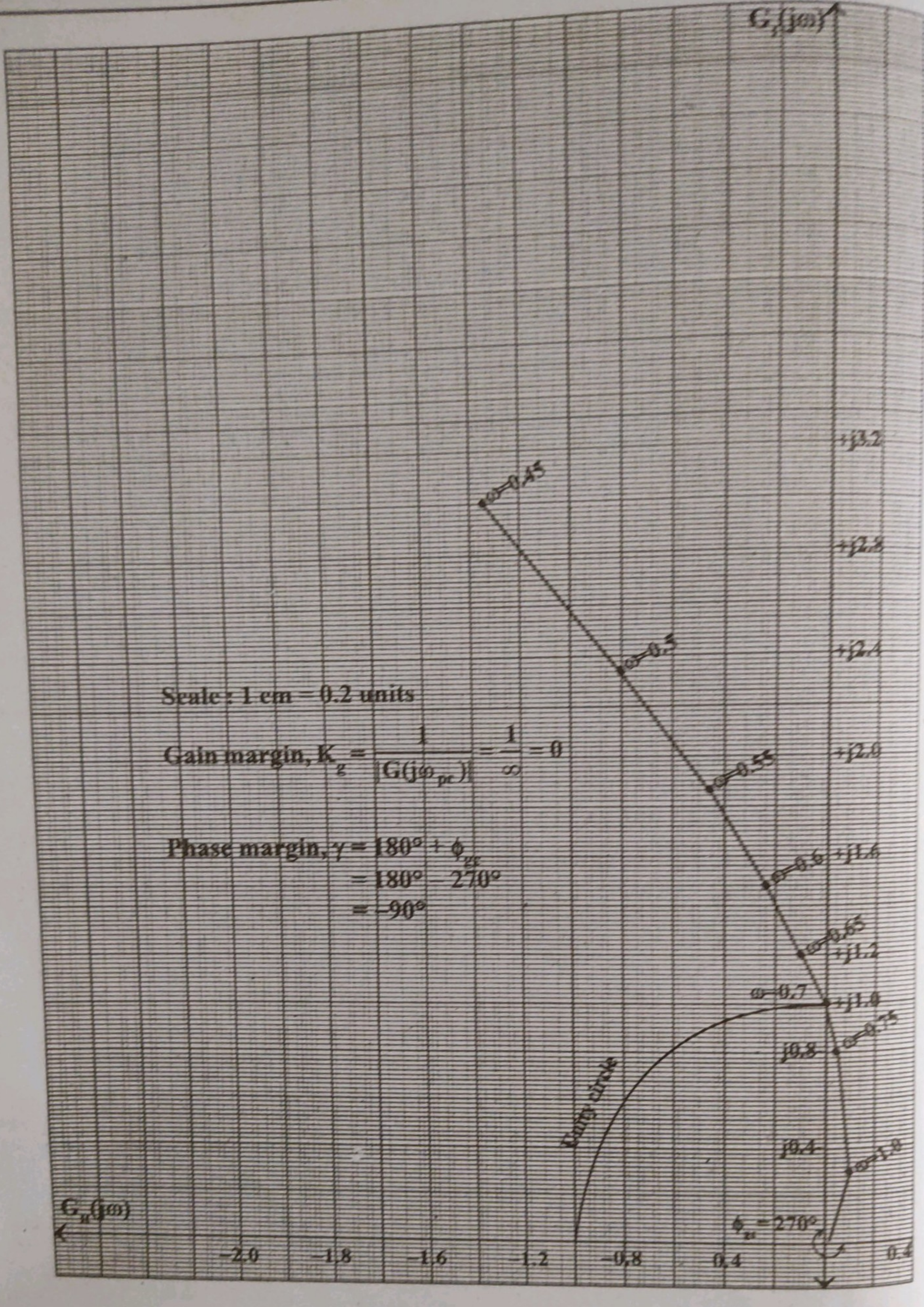


Fig 4.8.2: Polar plot of $G(j\omega) = 1/(j\omega)^2\,(1+j\omega)\,(1+j2\omega)$, (using rectangular coordinates)

EXAMPLE 4.10

The open loop transfer function of a unity feedback system is given by $G(s) = 1/s(1+s)^2$. Sketch polar plot and determine the gain and phase margin.

SOLUTION

Given that, $G(s) = 1/s(1+s)^2$.

Put $s = j\omega$,

$$\therefore G(j\omega) = \frac{1}{j\omega\,(1+j\omega)^2} = \frac{1}{j\omega\,(1+j\omega)\,(1+j\omega)}$$

The corner frequency is $\omega_{c1} = 1$ rad/sec. The magnitude and phase angle of $G(j\omega)$ are calculated corner frequency and frequencies around corner frequency and tabulated in table-1. Using pola rectangular conversion the polar coordinates listed in table-1 are converted to rectangular coordinates tabulated in table-2. The polar plot using polar coordinates is sketched on a polar graph sheet as show fig 4.10.1. The polar plot using rectangular coordinates are sketched on an ordinary graph sheet as sho in fig 4.10.2.

$$G(j\omega) = \frac{1}{j\omega\,(1+j\omega)^2} = \frac{1}{j\omega\,(1+j\omega)\,(1+j\omega)}$$

$$= \frac{1}{\omega\angle 90^\circ\ \sqrt{1+\omega^2}\ \angle\tan^{-1}\omega\ \sqrt{1+\omega^2}\ \angle\tan^{-1}\omega}$$

$$= \frac{1}{\omega\left(\sqrt{1+\omega^2}\right)^2}\quad \angle(-90^\circ - 2\tan^{-1}\omega)$$

$$|G(j\omega)| = \frac{1}{\omega(1+\omega^2)} = \frac{1}{\omega+\omega^3}$$

$$\angle G(j\omega) = -90^\circ - 2\tan^{-1}\omega$$

TABLE-1: Magnitude and phase of G(jw) at various frequencies

ω rad/sec	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1
|G(jω)|	2.2	1.6	1.2	1	0.8	0.6	0.5	0.4
∠G(jω) deg	−134	−143	−151	−159	−167	−174	−180	−185

TABLE-2 : Real and imaginary part of G(jω) at various frequencies

ω rad/sec	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1
$G_R(j\omega)$	−1.53	−1.28	−1.05	−0.93	−0.78	−0.6	−0.5	−0.4
$G_I(j\omega)$	−1.58	−0.96	−0.58	−0.36	−0.18	0.06	0	0.03

RESULT

Gain margin, $K_g = 2$

Phase margin, $\gamma = 21°$

EXAMPLE 4.11

Consider a unity feedback system having an open loop transfer function $G(s) = \frac{K}{s(1+0.2s)(1+0.05s)}$. Sketch the polar plot and determine the value of K so that (i) Gain margin is 18 db (ii) Phase margin is 60°.

SOLUTION

Given that, $G(s) = \frac{K}{s(1+0.2s)(1+0.05s)}$. The polar plot is sketched by taking K = 1.

$\therefore$ Put K = 1 and s = jω in G(s). $\therefore G(j\omega) = \frac{1}{j\omega(1+j0.2\omega)(1+j0.05\omega)}$

The corner frequencies are ω_{c1} = 1/0.2 = 5 rad/sec and ω_{c2} = 1/0.05 = 20 rad/sec. The magnitude and phase angle of G(jω) are calculated for various frequencies and tabulated in table-1. Using polar to rectangular conversion the polar coordinates listed in table-1 are converted to rectangular coordinates and tabulated in table-2. The polar plot using polar coordinates is sketched on a polar graph sheet as shown in fig 4.11.1. Polar plot using rectangular coordinates is sketched on an ordinary graph sheet as shown in fig 4.11.2.

$$G(j\omega) = \frac{1}{j\omega(1+j0.2\omega)(1+j0.05\omega)}$$

$$= \frac{1}{\omega\angle 90^\circ \sqrt{1+(0.2\omega)^2}\ \angle\tan^{-1}0.2\omega\ \sqrt{1+(0.05\omega)^2}\ \angle\tan^{-1}0.05\omega}$$

$$= \frac{1}{\omega\sqrt{1+(0.2\omega)^2}\ \sqrt{1+(0.05\omega)^2}}\ \angle(-90^\circ - \tan^{-1}0.2\omega - \tan^{-1}0.05\omega)$$

$$\therefore |G(j\omega)| = \frac{1}{\omega\sqrt{1+(0.2\omega)^2}\ \sqrt{1+(0.05\omega)^2}} \quad \text{and} \quad \angle G(j\omega) = -90^\circ - \tan^{-1}0.2\omega - \tan^{-1}0.05\omega$$

TABLE-1 : Magnitude and phase of G(jω) at various frequencies

ω rad/sec	0.6	0.8	1	2	3	4
|G(jω)|	1.65	1.23	1.0	0.5	0.3	0.2
∠G(jω) deg	−98	−101	−104	−117.5	−129.4	−140

ω rad/sec	5	6	7	9	10	11	14
|G(jω)|	0.14	0.1	0.07	0.05	0.04	0.03	0.02
∠G(jω) deg	−149	−157	−164	−176	−180	−184	−195

TABLE-2 : Real and imaginary part of G(jω) at various frequencies

ω rad/sec	0.6	0.8	1	2	3	4
$G_R(j\omega)$	−0.23	−0.23	−0.24	−0.23	−0.19	−0.15
$G_I(j\omega)$	−1.63	−1.21	−0.97	−0.44	−0.23	−0.13

EXAMPLE 4.12

Consider a unity feedback system having an open loop transfer function, $G(s) = \frac{K}{s\,(1+0.5s)\,(1+4s)}$. Sketch the polar plot and determine the value of K so that (i) Gain margin is 20 db and (ii) Phase margin is 30°.

SOLUTION

Given that, G(s) = K/s (1+0.5s) (1+4s)

The polar plot is sketched by taking K =1.

Put K = 1 and s = jω in G(s).

$$\therefore G(j\omega) = \frac{1}{j\omega\,(1+j0.5\omega)\,(1+j4\omega)}$$

The corner frequencies are ω_{c1} = 1/4 = 0.25 rad/sec and ω_{c2} = 1/0.5 = 2 rad/sec. The magnitude and phase angle of G(jω) are calculated for various frequencies and tabulated in table-1. Using polar to rectangular conversion the polar coordinates listed in table-1 are converted to rectangular coordinates and tabulated in table-2. The polar plot using polar coordinates is sketched on a polar graph sheet as shown in fig 4.12.1. The polar plot using rectangular coordinates is sketched on an ordinary graph sheet as shown in fig 4.12.2.

$$G(j\omega) = \frac{1}{j\omega\,(1+j0.5\omega)\,(1+j4\omega)}$$

$$= \frac{1}{\omega\angle 90^\circ\ \sqrt{1+(0.5\omega)^2}\ \angle \tan^{-1}0.5\omega\ \sqrt{1+(4\omega)^2}\ \angle\tan^{-1}4\omega}$$

$$= \frac{1}{\omega\sqrt{1+0.25\omega^2}\ \sqrt{1+16\omega^2}}\ \angle(-90^\circ - \tan^{-1}0.5\omega - \tan^{-1}4\omega)$$

$$\therefore |G(j\omega)| = \frac{1}{\omega\ \sqrt{1+0.25\omega^2}\ \sqrt{1+16\omega^2}}$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1}0.5\omega - \tan^{-1}4\omega$$

TABLE-1 : Magnitude and phase of G(jω) at various frequencies

ω rad/sec	0.3	0.4	0.5	0.6	0.8	1.0	1.2
|G(jω)|	2.11	1.3	0.87	0.61	0.35	0.22	0.15
∠G(jω) deg	−149	−159	−167	−174	−184	−193	−199

TABLE-2 : Real part and imaginary part of G(jω) at various frequencies

ω rad/sec	0.3	0.4	0.5	0.6	0.8	1.0	1.2
$G_R(j\omega)$	−1.8	−1.21	−0.85	−0.61	−0.35	−0.21	−0.14
$G_I(j\omega)$	−1.09	−0.47	−0.2	−0.06	0.02	0.05	0.05

From the polar plot, with K = 1,

Gain margin, $K_g = 1/0.44 = 2.27$

Gain margin in db = 20 log 2.27 = 7.12 db

Phase margin, $\gamma = 180° + \phi_{gc} = 180° - 165° = 15°$

Case (i)

With K = 1, let G(jω) cut the –180° axis at point B and gain corresponding to that point be G_B. Fr the polar plot, $G_B = 0.44$. The gain margin of 7.12 db with K = 1 has to be increased to 20 db and so K h to be decreased to a value less than one.

Let G_A be the gain at –180° for a gain margin of 20 db.

$$\text{Now, } 20\log\frac{1}{G_A} = 20$$

$$\log\frac{1}{G_A} = \frac{20}{20} = 1$$

$$\frac{1}{G_A} = 10^1 = 10$$

$$\therefore G_A = \frac{1}{10} = 0.1$$

The value of K is given by, $K = \frac{G_A}{G_B} = \frac{0.1}{0.44} = 0.227$

Case (ii)

With K = 1, the phase margin is 15°. This has to be increased to 30°. Hence the gain has to be decreased.

Let ϕ_{gc2} be the phase of G(jω) for a phase margin of 30°.

$$\therefore 30° = 180° + \phi_{gc2}$$

$$\phi_{gc2} = 30° - 180° = -150°$$

In the polar plot the –150° line cuts the locus of G(jω) at point C and cut the unity circle at point D.

Let, G_C = Magnitude of G(jω) at point C.

G_D = Magnitude of G(jω) at point D.

From the polar plot, $G_C = 2.04$ and $G_D = 1$

$$\text{Now, } K = \frac{G_D}{G_C} = \frac{1}{2.04} = 0.49$$

RESULT

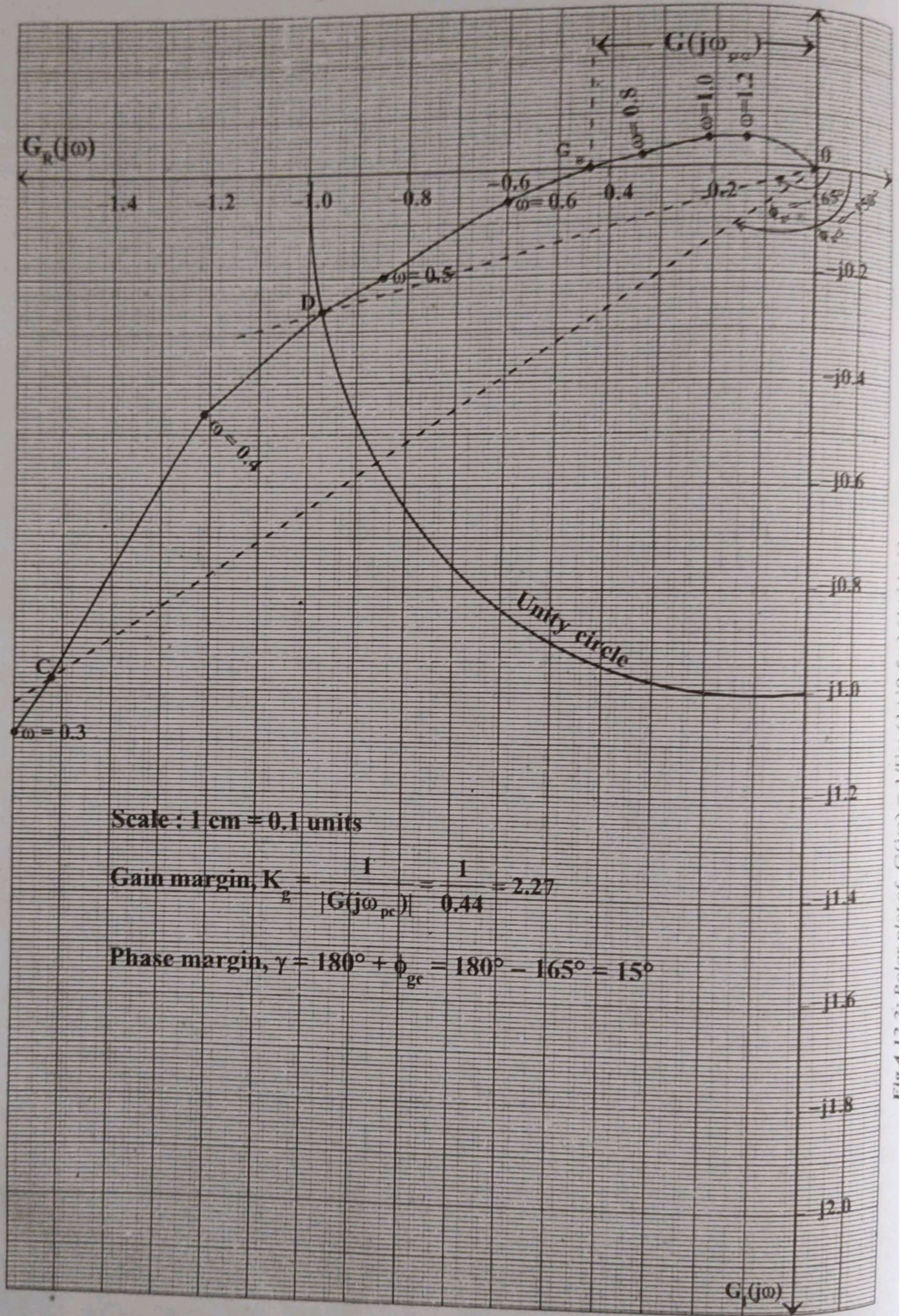
(i) When K = 1, Gain margin, $K_g = 2.27$

Gain margin in db = 7.12 db

(ii) When K = 1, Phase margin, $\gamma = 15°$

(iii) For a gain margin of 20 db, K = 0.227

(iv) For a phase margin of 30°, K = 0.49


G(jω_pc)
ω=0.8
ω=1.0
ω=1.2
G_R(jω)
G_e
0
−1.4
−1.2
−1.0
−0.8
−0.6
0.4
−0.2
ω=0.6
165°
ω=0.5
D
ω=0.4
C
ω = 0.3
−j0.2
−j0.4
−j0.6
−j0.8
−j1.0
−j1.2
−j1.4
−j1.6
−j1.8
−j2.0
Unity circle
Scale : 1 cm = 0.1 units
Gain margin, K_g = 1/|G(jω_pc)| = 1/0.44 = 2.27
Phase margin, γ = 180° + φ_gc = 180° − 165° = 15°
G_I(jω)