Given that, $R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$

$$\therefore r(t) = \mathcal{L}^{-1}\{R(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}\right\} = \frac{1}{3} - 2t + \frac{1}{3}\frac{t^2}{2!} = \frac{1}{3} - 2t + \frac{t^2}{6}$$

$$\dot{r}(t) = \frac{d}{dt}r(t) = -2 + \frac{1}{6}2t = -2 + \frac{t}{3}$$

$$\ddot{r}(t) = \frac{d^2}{dt^2}r(t) = \frac{d}{dt}\dot{r}(t) = \frac{1}{3}$$

$$\dddot{r}(t) = \frac{d^3}{dt^3}r(t) = \frac{d}{dt}\ddot{r}(t) = 0$$

$\therefore$ Error signal in time domain , $e(t) = \frac{1}{20}\frac{d}{dt}\ddot{r}(t) = \frac{1}{20}\left(\frac{1}{3}\right) = \frac{1}{60}$

Steady state error, $e_{ss} = \underset{t\to\infty}{\text{Lt}}\ e(t) = \underset{t\to\infty}{\text{Lt}}\ \frac{1}{60} = \frac{1}{60}$

## RESULT

a) Position error constant, $K_p = \infty$

Velocity error constant, $K_v = \infty$

Acceleration error constant, $K_a = 20$

b) When, $R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$, Steady state error, $e_{ss} = \frac{1}{60}$

## EXAMPLE 3.12

For servomechanisms with open loop transfer function given below explain what type of input signal give rise to a constant steady state error and calculate their values.

(i) $G(s) = \frac{20(s+2)}{s(s+1)(s+3)}$; (ii) $G(s) = \frac{10}{(s+2)(s+3)}$; (iii) $G(s) = \frac{10}{s^2(s+1)(s+2)}$

## SOLUTION

(i) $G(s) = \frac{20(s+2)}{s(s+1)(s+3)}$

Let us assume unity feedback system, $\therefore H(s)=1$

The open loop system has a pole at origin. Hence it is a type-1 system. In systems with type number-1, the velocity (ramp) input will give a constant steady state error.

The steady state error with unit velocity input, $e_{ss} = \frac{1}{K_v}$

Velocity error constant, $K_V = \underset{s\to 0}{\text{Lt}}\ s\,G(s)\,H(s) = \underset{s\to 0}{\text{Lt}}\ s\,G(s)$

$$= \underset{s\to 0}{\text{Lt}}\ s\,\frac{20(s+2)}{s(s+1)(s+3)} = \frac{20\times 2}{1\times 3} = \frac{40}{3}$$

Steady state error, $e_{SS} = \frac{1}{K_V} = \frac{3}{40} = 0.075$

(ii) $G(s)=\dfrac{10}{(s+2)(s+3)}$

Let us assume unity feedback system, $\therefore H(s)=1$.

The open loop system has no pole at origin. Hence it is a type-0 system. In systems with type number-0, the step input will give a constant steady state error.

The steady state error with unit step input, $e_{ss}=\dfrac{1}{1+K_p}$

Position error constant, $K_p=\underset{s\to 0}{\mathrm{Lt}}\ G(s)H(s)=\underset{s\to 0}{\mathrm{Lt}}\ G(s)=\underset{s\to 0}{\mathrm{Lt}}\ \dfrac{10}{(s+2)(s+3)}=\dfrac{10}{2\times 3}=\dfrac{5}{3}$

Steady state error, $e_{ss}=\dfrac{1}{1+K_p}=\dfrac{1}{1+\frac{5}{3}}=\dfrac{3}{3+5}=\dfrac{3}{8}=0.375$

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(iii) $G(s)=\dfrac{10}{s^2(s+1)(s+2)}$

Let us assume unity feedback system, $\therefore H(s)=1$.

The open loop system has two poles at origin. Hence it is a type-2 system. In systems with type number-2, the acceleration (parabolic) input will give a constant steady state error.

The steady state error with unit acceleration input, $e_{ss}=\dfrac{1}{K_a}$

Acceleration error constant, $K_a=\underset{s\to 0}{\mathrm{Lt}}\ s^2\,G(s)H(s)=\underset{s\to 0}{\mathrm{Lt}}\ s^2\,G(s)=\underset{s\to 0}{\mathrm{Lt}}\ s^2\dfrac{10}{s^2(s+1)(s+2)}=\dfrac{10}{1\times 2}=5$

Steady state error, $e_{ss}=\dfrac{1}{K_a}=\dfrac{1}{5}=0.2$

**RESULT**

1. In system (i) with unit velocity input, Steady state error $=0.075$
2. In system (ii) with unit step input, Steady state error $=0.375$
3. In system (iii) with unit acceleration input, Steady state error $=0.2$

## EXAMPLE 3.13

The open loop transfer function of a servo system with unity feedback is $G(s)=10/s(0.1s+1)$. Evaluate the static error constants of the system. Obtain the steady state error of the system. When subjected to an input given by the polynominal, $r(t)=a_0+a_1t+\dfrac{a_2}{2}t^2$.

**SOLUTION**

**To find static error constant**

For unity feedback system, $H(s)=1$.

$\therefore$ Loop transfer function, $G(s)\,H(s)=G(s)$

The static error constants are $K_p$, $K_v$ and $K_a$.

Position error constant, $K_p = \underset{s\to 0}{Lt}\ G(s) = \underset{s\to 0}{Lt}\ \frac{10}{s(0.1s+1)} = \infty$

Velocity error constant, $K_v = \underset{s\to 0}{Lt}\ sG(s) = \underset{s\to 0}{Lt}\ s\frac{10}{s(0.1s+1)} = 10$

Acceleration error constant, $K_a = \underset{s\to 0}{Lt}\ s^2G(s) = \underset{s\to 0}{Lt}\ s^2\frac{10}{s(0.1s+1)} = 0$

**To find steady state error**

## Method - I

Steady state error for non-standard input is obtained using generalized error series, given below.

The error signal, $e(t) = r(t)C_o + \dot{r}(t)C_1 + \ddot{r}(t)\frac{C_2}{2!} + .... + \overset{n.}{r}(t)\frac{C_n}{n!} + ....$

Given that, $r(t) = a_o + a_1t + \frac{a_2}{2}t^2$

$$\therefore\ \dot{r}(t) = \frac{d}{dt}r(t) = \frac{d}{dt}\left(a_o + a_1t + \frac{a_2}{2}t^2\right) = a_1 + a_2t$$

$$\ddot{r}(t) = \frac{d^2}{dt^2}r(t) = \frac{d}{dt}\left(\frac{d}{dt}r(t)\right) = \frac{d}{dt}(a_1 + a_2t) = a_2$$

$$\dddot{r}(t) = \frac{d^3}{dt^3}r(t) = \frac{d}{dt}\left(\frac{d^2}{dt^2}r(t)\right) = \frac{d}{dt}(a_2) = 0$$

Derivatives of r(t) is zero after 2nd derivative. Hence, let us evaluate three constants $C_o, C_1$ & $C_2$.

The generalized error constants are given by,

$$C_o = \underset{s\to 0}{Lt}\ F(s)\ ; \qquad C_1 = \underset{s\to 0}{Lt}\ \frac{d}{ds}F(s)\ ; \qquad C_2 = \underset{s\to 0}{Lt}\ \frac{d^2}{ds^2}F(s)$$

$$F(s) = \frac{1}{1+G(s)H(s)} = \frac{1}{1+G(s)} = \frac{1}{1+\dfrac{10}{s(0.1s+1)}} = \frac{s(0.1s+1)}{s(0.1s+1)+10} = \frac{0.1\,s^2+s}{0.1s^2+s+10}$$

$$C_o = \underset{s\to 0}{Lt}\ F(s) = \underset{s\to 0}{Lt}\ \frac{0.1s^2+s}{0.1s^2+s+10} = 0$$

$$C_1 = \underset{s\to 0}{Lt}\ \frac{d}{ds}F(s) = \underset{s\to 0}{Lt}\ \frac{d}{ds}\left[\frac{0.1s^2+s}{0.1s^2+s+10}\right]$$

$$= \underset{s\to 0}{Lt}\left[\frac{(0.1s^2+s+10)\,(0.2s+1) - (0.1s^2+s)\,(0.2s+1)}{(0.1s^2+s+10)^2}\right] = \underset{s\to 0}{Lt}\ \frac{2s+10}{(0.1s^2+s+10)^2} = \frac{10}{10^2} = 0.1$$

$$C_2 = \underset{s\to 0}{Lt}\ \frac{d^2}{ds^2}F(s) = \underset{s\to 0}{Lt}\ \frac{d}{ds}\left[\frac{d}{ds}F(s)\right] = \underset{s\to 0}{Lt}\ \frac{d}{ds}\left[\frac{2s+10}{(0.1s^2+s+10)^2}\right]$$

$$= \underset{s\to 0}{Lt}\ \frac{d}{ds}\left[\frac{(0.1s^2+s+10)^2 \times 2 - (2s+10) \times 2\,(0.1s^2+s+10)\,(0.2s+1)}{(0.1s^2+s+10)}\right]$$

$$\therefore C_2 = \frac{10^2 \times 2 - 10 \times 2 \times 10 \times 1}{10^4} = 0$$

Error signal, $e(t) = r(t)C_0 + \dot{r}(t)C_1 + \ddot{r}(t)\frac{C_2}{2!} = \dot{r}(t)C_1 + 0 + 0 = (a_1 + a_2 t)\ 0.1$

$\therefore$ Steady state error, $e_{ss} = \underset{t\to\infty}{Lt}\ e(t) = \underset{t\to\infty}{Lt}\left[\,(a_1 + a_2 t)\ 0.1\,\right] = \infty$

## Method - II

The error signal in s-domain, $E(s) = \dfrac{R(s)}{1 + G(s)H(s)}$

Given that, $r(t) = a_o + a_1 t + \dfrac{a_2}{2}t^2$; $\quad G(s) = \dfrac{10}{s(0.1s+1)}$; $\quad H(s) = 1$

On taking Laplace transform of r(t) we get R(s),

$$\therefore R(s) = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{2}\frac{2!}{s^3} = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3}$$

$$\therefore E(s) = \frac{R(s)}{1+G(s)H(s)} = \frac{\frac{a_o}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3}}{1 + \frac{10}{s\,(0.1s+1)}} = \frac{\frac{a_o}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3}}{\frac{s\,(0.1s+1)+10}{s\,(0.1s+1)}}$$

$$= \frac{a_0}{s}\left[\frac{s\,(0.1s+1)}{s\,(0.1s+1)+10}\right] + \frac{a_1}{s^2}\left[\frac{s\,(0.1s+1)}{s\,(0.1s+1)+10}\right] + \frac{a_2}{s^3}\left[\frac{s\,(0.1s+1)}{s\,(0.1s+1)+10}\right]$$

The steady state error $e_{ss}$ can be obtained from final value theorem.

Steady state error, $e_{ss} = \underset{t\to\infty}{Lt}\ e(t) = \underset{s\to 0}{Lt}\ s\,E(s)$

$$\therefore e_{ss} = \underset{s\to 0}{Lt}\ s\left\{\frac{a_o}{s}\left[\frac{s(0.1s+1)}{s(0.1s+1)+10}\right] + \frac{a_1}{s^2}\left[\frac{s(0.1s+1)}{s(0.1s+1)+10}\right] + \frac{a_2}{s^3}\left[\frac{s(0.1s+1)}{s(0.1s+1)+10}\right]\right\}$$

$$= \underset{s\to 0}{Lt}\left\{\frac{a_o s(0.1s+1)}{s(0.1s+1)+10} + \frac{a_1(0.1s+1)}{s(0.1s+1)+10} + \frac{a_2(0.1s+1)}{s\left[s(0.1s+1)+10\right]}\right\} = 0 + \frac{a_1}{10} + \infty = \infty$$

## Method - III

Error signal in s-domain, $E(s) = \dfrac{R(s)}{1+G(s)H(s)}$; $\qquad \therefore \dfrac{E(s)}{R(s)} = \dfrac{1}{1+G(s)H(s)}$

Given that, $G(s) = \dfrac{10}{s(0.1s+1)}$ and $H(s) = 1$.

$$\therefore \frac{E(s)}{R(s)} = \frac{1}{1 + \frac{10}{s(0.1s+1)}} = \frac{s(0.1s+1)}{s(0.1s+1)+10} = \frac{0.1s^2 + s}{0.1s^2 + s + 10} = \frac{s + 0.1s^2}{10 + s + 0.1s^2} = \frac{s}{10} - \frac{s^3}{1000} + \ldots.$$

$$\therefore E(s) = \frac{s}{10}R(s) - \frac{s^3}{1000}R(s) + \ldots$$

Dividing numerator polynomial by denominator polynomial.

$$\begin{array}{r|l}
 & \frac{s}{10}-\frac{s^3}{1000} \\ \hline
10+s+0.1s^2 & s+0.1s^2 \\
 & \underline{\underset{(-)}{}s\underset{(-)}{+}\frac{s^2}{10}\underset{(-)}{+}\frac{s^3}{100}} \\
 & \qquad -\frac{s^3}{100} \\
 & \qquad \underline{\underset{(+)}{-}\frac{s^3}{100}\underset{(+)}{-}\frac{s^4}{1000}\underset{(+)}{-}\frac{s^5}{10000}} \\
 & \qquad\qquad \frac{s^4}{1000}+\frac{s^5}{10000}
\end{array}$$

On taking inverse Laplace transform,

$$e(t)=\frac{1}{10}\dot{r}-\frac{1}{1000}\ddot{r}(t)+.....$$

Given that, $r(t)=a_0+a_1t+\frac{a_2}{2}t^2$

$$\therefore\quad \dot{r}=\frac{d}{dt}r(t)=a_1+a_2t$$

$$\ddot{r}(t)=\frac{d}{dt}\dot{r}(t)=a_2$$

$$\dddot{r}(t)=\frac{d}{dt}\ddot{r}(t)=0$$

$\therefore$ Error signal in time domain, $e(t)=\frac{1}{10}\dot{r}(t)=\frac{1}{10}(a_1+a_2t)$

Steady state error, $\quad e_{ss}=\underset{t\to\infty}{Lt}\ e(t)=\underset{t\to\infty}{Lt}\ \frac{1}{10}(a_1+a_2t)=\infty$

## RESULT

1. Position error constant, $K_p=\infty$
2. Velocity error constant, $K_v=10$
3. Acceleration error constant, $K_a=0$
4. When input, $r(t)=a_0+a_1t+\frac{a_2t^2}{2}$, Steady state error, $e_{ss}=\infty$

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## EXAMPLE 3.14

Consider a unity feedback system with a closed loop transfer function $\frac{C(s)}{R(s)}=\frac{Ks+b}{s^2+as+b}$. Determine open loop transfer function G(s). Show that steady state error with unit ramp input is given by $\frac{(a-K)}{b}$.

## SOLUTION

For unity feedback system , H(s)=1

The closed loop transfer function, $M(s)=\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)H(s)}=\frac{G(s)}{1+G(s)}$

$$\therefore\frac{G(s)}{1+G(s)}=M(s)$$

On cross multiplication of the above equation we get,

$$G(s)=M(s)[1+G(s)]=M(s)+M(s)\,G(s)$$

$$\therefore G(s)-M(s)\,G(s)=M(s)\quad\Rightarrow\quad G(s)[1-M(s)]=M(s)\quad\Rightarrow\quad M(s)=\frac{Ks+b}{s^2+as+b}$$

$\therefore$ Open loop transfer function, $G(s)=\frac{M(s)}{1-M(s)}=\frac{\frac{Ks+b}{s^2+as+b}}{1-\frac{Ks+b}{s^2+as+b}}=\frac{Ks+b}{(s^2+as+b)-(Ks+b)}$

$$=\frac{Ks+b}{s^2+as+b-Ks-b}=\frac{Ks+b}{s^2+(a-K)s}=\frac{Ks+b}{s[s+(a-K)]}$$

Velocity error constant, $K_V = \underset{s\to 0}{Lt}\, s\,G(s)H(s) = \underset{s\to 0}{Lt}\, s\,G(s) = \underset{s\to 0}{Lt}\, s\frac{Ks+b}{s[s+(a-K)]} = \frac{b}{a-K}$

With velocity input, Steady state error, $e_{ss} = \frac{1}{K_V} = \frac{a-K}{b}$

**RESULT**

Open loop transfer function, $G(s) = \frac{Ks+b}{s\,[s+(a-K)]}$

With veloctiy input, Steady state error, $e_{ss} = \frac{a-K}{b}$

## EXAMPLE 3.15

A unity feedback system has the forward transfer function $G(s) = \frac{K_1(2s+1)}{s\,(5s+1)(1+s)^2}$. When the input $r(t) = 1+6t$, determine the minimum value of $K_1$ so that the steady error is less than 0.1.

**SOLUTION**

Given that, input $r(t) = 1+6t$

On taking laplace transform of r(t) we get R(s).

$$\therefore R(s) = \mathcal{L}\{r(t)\} = \mathcal{L}\{1+6t\} = \frac{1}{s}+\frac{6}{s^2}$$

The error signal in s-domain E(s) is given by,

$$\therefore E(s) = \frac{R(s)}{1+G(s)H(s)} = \frac{\frac{1}{s}+\frac{6}{s^2}}{1+\frac{K_1(2s+1)}{s(5s+1)(1+s)^2}} = \frac{\frac{1}{s}+\frac{6}{s^2}}{\frac{s\,(5s+1)\,(1+s)^2+K_1(2s+1)}{s\,(5s+1)\,(1+s)^2}}$$

Here H(s)=1

$$= \frac{1}{s}\left[\frac{s\,(5s+1)\,(1+s)^2}{s\,(5s+1)\,(1+s)^2+K_1(2s+1)}\right] + \frac{6}{s^2}\left[\frac{s\,(5s+1)\,(1+s)^2}{s\,(5s+1)\,(1+s)^2+K_1(2s+1)}\right]$$

The steady state error $e_{ss}$ can be obtained from final value theorem.

$$e_{ss} = \underset{t\to\infty}{Lt}\, e(t) = \underset{s\to 0}{Lt}\, s\,E(s)$$

$$= \underset{s\to 0}{Lt}\, s\left\{\frac{1}{s}\left[\frac{s(5s+1)(1+s)^2}{s(5s+1)(1+s)^2+K_1(2s+1)}\right] + \frac{6}{s^2}\left[\frac{s(5s+1)(1+s)^2}{s(5s+1)(1+s)^2+K_1(2s+1)}\right]\right\}$$

$$= \underset{s\to 0}{Lt}\left\{\frac{s(5s+1)(1+s)^2}{s(5s+1)(1+s)^2+K_1(2s+1)} + \frac{6(5s+1)(1+s)^2}{s(5s+1)(1+s)^2+K_1(2s+1)}\right\} = 0+\frac{6}{K_1} = \frac{6}{K_1}$$

Given that, $e_{ss} < 0.1$, $\therefore 0.1 = \frac{6}{K_1}$ or $K_1 = \frac{6}{0.1} = 60$

**RESULT**

For steady state error, $e_{ss} < 0.1$, the value of $K_1$ should be greater than 60.