+ to −, − to +, + to −

s^5	:	9	10	−9	.... Row-1
s^4	:	−20	−1	−10	.... Row-2
s^3	:	9.55	−13.5		.... Row-3
s^2	:	−29.3	−10		.... Row-4
s^1	:	−16.8			.... Row-5
s^0	:	−10			.... Row-6

Column-1

$$s^3 : \frac{-20\times10-(-1)\times9}{-20} \quad \frac{-20\times(-9)-(-10)\times9}{-20}$$

s^3: 9.55 −13.5

$$s^2 : \frac{9.55\times(-1)-(-13.5)\times(-20)}{9.55} \quad \frac{9.55\times(-10)}{9.55}$$

s^2: −29.3 −10

$$s^1 : \frac{-29.3\times(-13.5)-(-10)\times9.55}{-29.3}$$

s^1: −16.8

$$s^0 : \frac{-16.8\times(-10)}{-16.8}$$

s^0: −10

By examining the elements of Ist column of routh array it is observed that there are three sign changes and so three roots are lying on the right half of s-plane and the remaining two roots are lying on the left half of s-plane.

RESULT

1. The system is unstable.
2. Three roots are lying on right half of s-plane and two roots are lying on left half of s-plane.

EXAMPLE 5.5

The characteristic polynomial of a system is, $s^7+9s^6+24s^5+24s^4+24s^3+24s^2+23s+15=0$. Determine the location of roots on s-plane and hence the stability of the system.

SOLUTION

METHOD-I

The characteristic equation is, $s^7+9s^6+24s^5+24s^4+24s^3+24s^2+23s+15=0$.

The given characteristic polynomial is 7th order equation and so it has 7 roots. Since the highest power of s is odd number, form the first row of array using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s as shown below.

s^7	:	1	24	24	23	.... Row-1
s^6	:	9	24	24	15	.... Row-2

Divide s^6 row by 3 to simplify the computations.

s^7	:	1	24	24	23	.... Row-1
s^6	:	3	8	8	5	.... Row-2
s^5	:	1	1	1		.... Row-3
s^4	:	1	1	1		.... Row-4
s^3	:	0	0			.... Row-5
s^3	:	2	1			.... Row-5
s^2	:	0.5	1			.... Row-6
s^1	:	−3				.... Row-7
s^0	:	1				.... Row-8

+ to −, − to +

Column-1

$$s^5 : \frac{3\times24-8\times1}{3} \quad \frac{3\times24-8\times1}{3} \quad \frac{3\times23-5\times1}{3}$$

s^5 : 21.33 21.33 21.33

Divide by 21.33

s^5 : 1 1 1

$$s^4 : \frac{1\times8-1\times3}{1} \quad \frac{1\times8-1\times3}{1} \quad \frac{1\times5-0\times3}{1}$$

s^4 : 5 5 5

Divide by 5

s^5 : 1 1 1

$$s^3 : \frac{1\times1-1\times1}{1} \quad \frac{1\times1-1\times1}{1}$$

s^3 : 0 0

On examining the first column elements of routh array it is found that there are two sign changes. Hence two roots are lying on the right half of s-plane and so the system is unstable.

The row of all zeros indicate the possibility of roots on imaginary axis. This can be tested by evaluating the roots of auxiliary polynomial.

The auxiliary equation is, $s^4 + s^2 + 1 = 0$

Put, $s^2 = x$ in the auxiliary equation,

$$\therefore s^4 + s^2 + 1 = x^2 + x + 1 = 0$$

The roots of quadratic are, $x = \dfrac{-1 \pm \sqrt{1-4}}{2} = -\dfrac{1}{2} \pm j\dfrac{\sqrt{3}}{2}$

$$= 1\angle 120^\circ \text{ or } 1\angle -120^\circ$$

But $s^2 = x$, $\therefore s = \pm\sqrt{x} = \pm\sqrt{1\angle 120^\circ}$ or $\pm\sqrt{1\angle -120^\circ}$

$$= \pm\sqrt{1}\angle 120^\circ/2 \quad \text{or} \quad \pm\sqrt{1}\angle -120^\circ/2$$

$$= \pm 1\angle 60^\circ \quad \text{or} \quad \pm 1\angle -60^\circ$$

$$= \pm(0.5 + j0.866) \quad \text{or} \quad \pm(0.5 - j0.866)$$

The auxiliary polynomial is,

$$A = s^4 + s^2 + 1$$

Differentiate A with respect to s.

$$\frac{dA}{ds} = 4s^3 + 2s$$

$s^3 : 4 \quad 2$

Divide by 2

$s^3 : 2 \quad 1$

$s^2 : \dfrac{2\times1-1\times1}{2} \quad \dfrac{2\times1-0\times1}{2}$

$s^2 : 0.5 \quad 1$

$s^1 : \dfrac{0.5\times1-1\times2}{0.5}$

$s^1 : -3$

$s^0 : \dfrac{-3\times1}{-3}$

$s^0 : 1$

Two roots of auxiliary polynomial are lying on the right half of s-plane and the remaining two on t left half of s-plane. The roots of auxiliary equation are also the roots of characteristic polynomial. The t roots lying on the right half of s-plane are indicated by two sign changes in the first column of routh arra The remaining five roots are lying on the left half of s-plane. No roots are lying on imaginary axis.

RESULT

1. The system is unstable.
2. Two roots are lying on right half of s-plane and five roots are lying on left half of s-plane.

METHOD-II

The characteristic equation is, $s^7 + 9s^6 + 24s^5 + 24s^4 + 24s^3 + 24s^2 + 23s + 15 = 0$

The given characteristic polynomial is 7th order equation and so it has 7 roots. Since the highest power of s is odd number, form the first row of array using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s as shown below.

s^7	:	1	24	24	23	.... Row-1
s^6	:	9	24	24	15	.... Row-2

Divide s^6 row by 3 to simplify the computations.

s^7	:	1	24	24	23	.... Row-1
s^6	:	3	8	8	5	.... Row-2
s^5	:	1	1	1		.... Row-3
s^4	:	1	1	1		.... Row-4
s^3	:	0	0			.... Row-5

$s^5 : \dfrac{3\times24-8\times1}{3} \quad \dfrac{3\times24-8\times1}{3} \quad \dfrac{3\times23-5\times1}{3}$

$s^5 : 21.33 \quad 21.33 \quad 21.33$

Divide by 21.33

$s^5 : 1 \quad 1 \quad 1$

$s^4 : \dfrac{1\times8-1\times3}{1} \quad \dfrac{1\times8-1\times3}{1} \quad \dfrac{1\times5-0\times3}{1}$

$s^4 : 5 \quad 5 \quad 5$

Divide by 5

$s^4 : 1 \quad 1 \quad 1$

$s^3 : \dfrac{1\times1-1\times1}{1} \quad \dfrac{1\times1-1\times1}{1}$

$s^3 : 0 \quad 0$

Since we get a row of zeros, there exists an even polynomial, the even polynomial is nothing but, the auxiliary polynomial.

The auxiliary polynomial is,

$$s^4 + s^2 + 1 = 0$$

Divide the characteristic equation by auxiliary polynomial to get the quotient polynomial.

$$\begin{array}{r|l}
\begin{array}{c} s^4+s^2+1 \\ \text{(Even Polynomial)} \end{array} & s^7+9s^6+24s^5+24s^4+24s^3+24s^2+23s+15 \quad \left(s^3+9s^2+23s+15 \right. \text{ (Quotient Polynomial)} \\
 & \underline{\overset{(-)}{s^7} \quad \overset{(-)}{+s^5} \quad \overset{(-)}{+s^3}} \\
 & 9s^6+23s^5+24s^4+23s^3+24s^2+23s+15 \\
 & \underline{\overset{(-)}{9s^6} + \quad \overset{(-)}{+9s^4} \quad \overset{(-)}{+9s^2}} \\
 & 23s^5+15s^4+23s^3+15s^2+23s+15 \\
 & \underline{\overset{(-)}{23s^5} \quad \overset{(-)}{+23s^3} \quad \overset{(-)}{+23s}} \\
 & 15s^4 \quad +15s^2 \quad +15 \\
 & \underline{\overset{(-)}{15s^4} \quad \overset{(-)}{+15s^2} \quad \overset{(-)}{+15}} \\
 & 0
\end{array}$$

The characteristic polynomial can be expressed as a product of quotient polynomial and auxiliary polynomial.

$$\therefore s^7+9s^6+24s^5+24s^4+24s^3+24s^2+23s+15=0$$

$$\Downarrow$$

$$\underbrace{(s^4+s^2+1)}_{\text{Even polynomial}}\ \underbrace{(s^3+9s^2+23s+15)}_{\text{Quotient polynomial}}=0$$

The routh array is constructed for quotient polynomial as shown below.

s^3	:	1	23
s^2	:	9	15

Divide s^2 row by 3,

s^3	:	1	23
s^2	:	3	5
s^1	:	21.33	
s^0	:	5	

Column-1

$$s^1 : \frac{3\times 23 - 5\times 1}{3}$$

$$s^1 : 21.33$$

$$s^0 : \frac{21.33\times 5 - 0\times 3}{21.33}$$

$$s^0 : 5$$

The elements of column-1 of quotient polynomial are all positive and there is no sign change. Hence all the roots of quotient polynomial are lying on the left half of s-plane. To determine the stability the roots of auxiliary polynomial should be evaluated.

The auxiliary equation is, $s^4 + s^2 + 1 = 0$.

Put, $s^2 = x$ in the auxiliary equation. $\quad \therefore s^4+s^2+1 = x^2+x+1=0$

The roots of quadradic are, $x = \dfrac{-1\pm\sqrt{1-4}}{2} = -\dfrac{1}{2}\pm j\dfrac{\sqrt{3}}{2} = 1\angle 120^\circ$ or $1\angle -120^\circ$

But $s^2 = x$, $\quad \therefore s = \pm\sqrt{x}$

$$= \pm\sqrt{1\angle 120^\circ} \quad \text{or} \quad \pm\sqrt{1\angle -120^\circ}$$

$$= \pm\sqrt{1}\angle 120^\circ/2 \quad \text{or} \quad \pm\sqrt{1}\angle -120^\circ/2$$

$$= \pm 1\angle 60^\circ \quad \text{or} \quad \pm 1\angle -60^\circ$$

$$= \pm(0.5 + j0.866) \quad \text{or} \quad \pm(0.5 - j0.866)$$

The roots of auxiliary equation are complex and has quadrantal symmetry. Two roots of auxiliary equation are lying on the right half of s-plane and the other two on the left half of s-plane.

The roots of characteristic equation are given by the roots of auxiliary polynomial and the roots of quotient polynomial. Hence we can conclude that two roots of characteristic equation are lying on the right half of s-plane and so system is unstable. The remaining five roots are lying on left half of s-plane.

EXAMPLE 5.6

The characteristic polynomial of a system is $s^7+5s^6+9s^5+9s^4+4s^3+20s^2+36s+36=0$. Determ the location of roots on the s-plane and hence the stability of the system.

SOLUTION

The characteristic equation is, $s^7+5s^6+9s^5+9s^4+4s^3+20s^2+36s+36=0$.

The given characteristic polynomial is 7th order equation and so it has 7 roots. Since the high power of s is odd number, form the first row of array using the coefficients of odd powers of s and fo the second row of array using the coefficients of even powers of s as shown below.

s^7	:	1	9	4	36	.... Row-1
s^6	:	5	9	20	36	.... Row-2

Divide s^6 row by 5 to simplify the computations.

s^7	:	1	9	4	36	.... Row-1
s^6	:	1	1.8	4	7.2	.... Row-2
s^5	:	1	0	4		.... Row-3
s^4	:	1	0	4		.... Row-4
s^3	:	0	0			.... Row-5

$$s^5 : \frac{1\times 9-1.8\times 1}{1} \quad \frac{1\times 4-4\times 1}{1} \quad \frac{1\times 36-7.2\times 1}{1}$$

$$s^5 : 7.2 \quad 0 \quad 28.8$$

Divide by 7.2

$$s^5 : 1 \quad 0 \quad 4$$

$$s^4 : \frac{1\times 1.8-0\times 1}{1} \quad \frac{1\times 4-4\times 1}{1} \quad \frac{1\times 7.2-0\times 1}{1}$$

$$s^4 : 1.8 \quad 0 \quad 7.2$$

Divide by 1.8

$$s^4 : 1 \quad 0 \quad 4$$

$$s^3 : \frac{1\times 0-0\times 1}{1} \quad \frac{1\times 4-4\times 1}{1}$$

$$s^3 : 0 \quad 0$$

The row of all zeros indicate the existence of even polynomial, which is also the auxiliary polynomial. The auxiliary polynomial is, $s^4+4=0$. Divide the characteristic equation by auxiliary equation to get the quotient polynomial.

$$
\begin{array}{r|l}
s^4+4 & s^7+5s^6+9s^5+9s^4+4s^3+20s^2+36s+36 \quad (s^3+5s^2+9s+9 \\
 & s^7 \qquad\qquad\qquad\qquad +4s^3 \\
\hline
 & 5s^6+9s^5+9s^4+20s^2+36s+36 \\
 & 5s^6 \qquad\qquad\qquad +20s^2 \\
\hline
 & 9s^5+9s^4+36s+36 \\
 & 9s^5 \qquad\quad +36s \\
\hline
 & 9s^4+36 \\
 & 9s^4+36 \\
\hline
 & 0
\end{array}
$$

The characteristic equation can be expressed as a product of quotient polynomial and auxiliary equation.

$$\therefore s^7+5s^6+9s^5+9s^4+4s^3+20s^2+36s+36=0$$

$$\underbrace{(s^4+4)}_{\text{Even polynomial}} \quad \underbrace{(s^3+5s^2+9s+9)}_{\text{Quotient polynomial}}=0$$

The routh array is constructed for quotient polynomial as shown below.

s^3	:	1	9
s^2	:	5	9
s^1	:	7.2	
s^0	:	9	

Column-1

$$s^1 : \frac{5\times 9-9\times 1}{5}$$

$$s^1 : 7.2$$

$$s^0 : \frac{7.2\times 9-0\times 5}{7.2}$$

$$s^0 : 9$$

There is no sign change in the elements of first column of routh array of quotient polynomial. Hence all the roots of quotient polynomial are lying on the left half of s-plane.

To determine the stability, the roots of auxiliary polynomial should be evaluated.

The auxiliary polynomial is, $s^4+4=0$.

Put, $s^2 = x$ in the auxiliary equation, $\therefore\ s^4 + 4 = x^2 + 4 = 0$

$$\therefore\ x^2 = -4 \quad \Rightarrow \quad x = \pm\sqrt{-4} = \pm j2 = 2\angle 90^\circ \text{ or } 2\angle -90^\circ$$

$$\text{But, } s = \pm\sqrt{x} = \pm\sqrt{2\angle 90^\circ} \quad \text{or} \quad \pm\sqrt{2\angle -90^\circ} = \pm\sqrt{2}\angle 90^\circ/2 \quad \text{or} \quad \pm\sqrt{2}\angle -90^\circ/2$$

$$= \pm\sqrt{2}\angle 45^\circ \quad \text{or} \quad \pm\sqrt{2}\angle -45 = \pm(1+j1) \quad \text{or} \quad \pm(1-j1)$$

The roots of auxiliary equation are complex and has quadrantal symmetry. Two roots of auxiliary equation are lying on the right half of s-plane and the other two on the left half of s-plane.

The roots of characteristic equation are given by roots of quotient polynomial and auxiliary polynomial. Hence we can conclude that two roots of characteristic equation are lying on the right half of s-plane and so the system is unstable. The remaining five roots are lying on the left half of s-plane.

RESULT

1. The system is unstable.
2. Two roots are lying on the right half of s-plane and five roots are lying on the left half of s-plane.

EXAMPLE 5.7

Use the routh stability criterion to determine the location of roots on the s-plane and hence the stability for the system represented by the characteristic equation $s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4 = 0$.

SOLUTION

The characteristic equation of the system is, $s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4 = 0$.

The given characteristic polynomial is 5th order equation and so it has 5 roots. Since the highest power of s is odd number, form the first row of routh array using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s.

$$\begin{array}{llll} s^5 : & 1 & 8 & 7 & \text{.... Row-1} \\ s^4 : & 4 & 8 & 4 & \text{.... Row-2} \end{array}$$

Divide s^4 row by 4 to simplify the calculations.

$$\begin{array}{llll} s^5 : & 1 & 8 & 7 & \text{.... Row-1} \\ s^4 : & 1 & 2 & 1 & \text{.... Row-2} \\ s^3 : & 1 & 1 & & \text{.... Row-3} \\ s^2 : & 1 & 1 & & \text{.... Row-4} \\ s^1 : & \epsilon & & & \text{.... Row-5} \\ s^0 : & 1 & & & \text{.... Row-6} \end{array}$$

Column-1

$$s^3 : \frac{1\times 8 - 2\times 1}{1} \quad \frac{1\times 7 - 1\times 1}{1}$$

$$s^3 : 6 \quad 6$$

Divide by 6

$$s^3 : 1 \quad 1$$

$$s^2 : \frac{1\times 2 - 1\times 1}{1} \quad \frac{1\times 1 - 0\times 1}{1}$$

$$s^2 : 1 \quad 1$$

$$s^1 : \frac{1\times 1 - 1\times 1}{1}$$

$$s^1 : 0$$

Let $0 \rightarrow \epsilon$

$$s^1 : \epsilon$$

$$s^0 : \frac{\epsilon\times 1 - 0\times 1}{\epsilon}$$

$$s^0 : 1$$

When $\epsilon \rightarrow 0$, there is no sign change in the first column of routh array. But we have a row of all zeros (s^1 row or row-5) and so there is a possibility of roots on imaginary axis. This can be found from the roots of auxiliary polynomial. Here the auxiliary polynomial is given by s^2 row.

The auxiliary polynomial is, $s^2 + 1 = 0$; $\quad \therefore\ s^2 = -1$ or $s = \pm\sqrt{-1} = \pm j1$

The roots of auxiliary polynomial are +j1, and −j1, lying on imaginary axis. The roots of auxiliary polynomial are also roots of characteristic equation. Hence two roots of characteristic equation are lying on imaginary axis and so the system is limitedly or marginally stable. The remaining three roots of characteristic equation are lying on the left half of s-plane.

RESULT

1. The system is limitedly or marginally stable.
2. Two roots are lying on imaginary axis and three roots are lying on left half of s-plane.

EXAMPLE 5.8

Use the routh stability criterion to determine the location of roots on the s-plane and hence the stability for the system represented by the characteristic equation, $s^6 + s^5 + 3s^4 + 3s^3 + 3s^2 + 2s + 1 = 0$.

SOLUTION

The characteristic polynomial of the system is, $s^6 + s^5 + 3s^4 + 3s^3 + 3s^2 + 2s + 1 = 0$.

The given characteristic polynomial is 6th order equation and so it has 6 roots. Since the highest power of s is even number, form the first row of routh array using the coefficients of even powers of s and form the second row using the coefficients of odd powers of s as shown below.

$$
\begin{array}{llllll}
s^6 & : & 1 & 3 & 3 & 1 \quad \text{.... Row-1} \\
s^5 & : & 1 & 3 & 2 & \quad \text{.... Row-2} \\
s^4 & : & \epsilon & 1 & 1 & \quad \text{.... Row-3} \\
s^3 & : & \dfrac{3\epsilon-1}{\epsilon} & \dfrac{2\epsilon-1}{\epsilon} & & \quad \text{.... Row-4} \\
s^2 & : & \dfrac{-2\epsilon^2+4\epsilon-1}{3\epsilon-1} & 1 & & \quad \text{.... Row-5} \\
s^1 & : & \dfrac{4\epsilon^2-\epsilon}{2\epsilon^2-4\epsilon+1} & & & \quad \text{.... Row-6} \\
s^0 & : & 1 & & & \quad \text{.... Row-7}
\end{array}
$$

$$s^4 : \frac{1\times3-3\times1}{1} \quad \frac{1\times3-2\times1}{1} \quad \frac{1\times1-0\times1}{1}$$

$$s^4 : 0 \quad 1 \quad 1$$

let $0 \to \epsilon$

$$s^4 : \epsilon \quad 1 \quad 1$$

$$s^3 : \frac{\epsilon\times3-1\times1}{\epsilon} \quad \frac{\epsilon\times2-1\times1}{\epsilon}$$

$$s^3 : \frac{3\epsilon-1}{\epsilon} \quad \frac{2\epsilon-1}{\epsilon}$$

$$s^2 : \frac{\frac{3\epsilon-1}{\epsilon} - \frac{2\epsilon-1}{\epsilon}\times\epsilon}{\frac{3\epsilon-1}{\epsilon}} \quad \frac{\frac{3\epsilon-1}{\epsilon}\times1-0\times}{\frac{3\epsilon-1}{\epsilon}}$$

$$s^2 : \frac{-2\epsilon^2+4\epsilon-1}{3\epsilon-1} \quad 1$$

$$s^1 : \frac{\frac{-2\epsilon^2+4\epsilon-1}{3\epsilon-1}\times\frac{2\epsilon-1}{\epsilon} - \frac{3\epsilon-1}{\epsilon}\times1}{\frac{-2\epsilon^2+4\epsilon-1}{3\epsilon-1}}$$

$$s^1 : \frac{(-2\epsilon^2+4\epsilon-1)(2\epsilon-1)-(3\epsilon-1)(3\epsilon}{\epsilon(-2\epsilon^2+4\epsilon-1)}$$

$$s^1 : \frac{-4\epsilon^3+\epsilon^2}{\epsilon(-2\epsilon^2+4\epsilon-1)} = \frac{4\epsilon^2-\epsilon}{2\epsilon^2-4\epsilon+1}$$

$$s^0 : \frac{\frac{4\epsilon^2-\epsilon}{4\epsilon^2-4\epsilon+1}\times1-0\times\frac{-2\epsilon^2+4\epsilon-1}{3\epsilon-1}}{(4\epsilon^2-\epsilon)/(4\epsilon^2-4\epsilon+1)}$$

$$s^0 : 1$$

On letting $\epsilon \to 0$, we get,

$$
\begin{array}{llllll}
s^6 & : & 1 & 3 & 3 & 1 \quad \text{.... Row-1} \\
s^5 & : & 1 & 3 & 2 & \quad \text{.... Row-2} \\
s^4 & : & 0 & 1 & 1 & \quad \text{.... Row-3} \\
s^3 & : & -\infty & -\infty & & \quad \text{.... Row-4} \\
s^2 & : & 1 & 1 & & \quad \text{.... Row-5} \\
s^1 & : & 0 & & & \quad \text{.... Row-6} \\
s^0 & : & 1 & & & \quad \text{.... Row-7}
\end{array}
$$

+ to −

− to +

Since there is a row of all zeros (s^1 row) there is a possibility of roots on imaginary axis. The auxiliary polynomial is $s^2 + 1 = 0$.

The roots of auxiliarypolynomial are, $s = \pm\sqrt{-1} = \pm j1$

The roots of auxiliary polynomial are also roots of characteristic equation. Hence two roots are lying on imaginary axis. Therefore divide the characteristic polynomial by auxiliary equation and construct the routh array for quotient polynomial to find the roots lying on right half of s-plane.

The characteristic polynomial can be expressed as a product of auxiliary polynomial and quotient polynomial.

$$\therefore s^6+s^5+3s^4+3s^3+3s^2+2s+1=0 \Rightarrow \underbrace{(s^2+1)}_{\text{Even polynomial}}\ \underbrace{(s^4+s^3+2s^2+2s+1)}_{\text{Quotient polynomial}}=0$$

The routh array for quotient polynomial is constructed as shown below.

$$\begin{array}{lllll} s^4 & : & 1 & 2 & 1 & \text{.... Row-1}\\ s^3 & : & 1 & 2 & & \text{.... Row-2}\\ s^2 & : & \epsilon & 1 & & \text{.... Row-3}\\ s^1 & : & \dfrac{2\epsilon-1}{\epsilon} & & & \text{.... Row-4}\\ s^0 & : & 1 & & & \text{.... Row-5} \end{array}$$

On letting $\epsilon \to 0$, we get

$$\begin{array}{lllll} s^4 & : & 1 & 2 & 1 & \text{.... Row-1}\\ s^3 & : & 1 & 2 & & \text{.... Row-2}\\ s^2 & : & 0 & 1 & & \text{.... Row-3}\\ s^1 & : & -\infty & & & \text{.... Row-4}\\ s^0 & : & 1 & & & \text{.... Row-5} \end{array}$$

Column-1

$$\begin{array}{r|l} & s^4+s^3+2s^2+2s+1 \quad \text{(Quotient polynomial)}\\ \hline (\text{Even polynomial})\ s^2+1 & s^6+s^5+3s^4+3s^3+3s^2+2s+1\\ & \underline{s^6 \overset{(-)}{} \quad \overset{+}{\underset{(-)}{}}\ s^4}\\ & s^5+2s^4+3s^3+3s^2+2s+1\\ & \underline{(-)s^5 \quad (-)+s^3}\\ & 2s^4+2s^3+3s^2+2s+1\\ & \underline{(-)2s^4 \quad (-)+2s^2}\\ & 2s^3+s^2+2s+1\\ & \underline{(-)2s^3 \quad (-)+2s}\\ & s^2 \quad +1\\ & \underline{(-)s^2 \quad (-)+1}\\ & 0 \end{array}$$

$$s^2 : \frac{1\times2-2\times1}{1} \quad \frac{1\times1-0\times1}{1}$$

$$s^2 : 0 \qquad 1$$

let $0 \to \epsilon$

$$s^2 : \epsilon \quad 1$$

$$s^1 : \frac{\epsilon\times2-1\times1}{\epsilon}$$

$$s^1 : \frac{2\epsilon-1}{\epsilon}$$

$$s^0 : \frac{\frac{2\epsilon-1}{\epsilon}\times1-0\times\epsilon}{(2\epsilon-1)/\epsilon}$$

$$s^0 : 1$$

On examining the first column of the routh array of quotient polynomial, we found that there are two sign changes. Hence two roots are lying on the right half of s-plane and other two roots of quotient polynomial are lying on the left half of s-plane.

The roots of characteristic equation are given by roots of auxiliary polynomial and quotient polynomial. Hence two roots are lying on imaginary axis, two roots are lying on right half of s-plane and the remaining two roots are lying on left half of s-plane. Hence the system is unstable.

RESULT

1. The system is unstable.
2. Two roots are lying on imaginary axis, two roots are lying on right half of s-plane and two roots are lying on left half of s-plane.

EXAMPLE 5.9

Determine the range of K for stability of unity feedback system whose open loop transfer fun

is $G(s) = \dfrac{K}{s(s+1)(s+2)}$.

SOLUTION

The closed loop transfer function, $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)} = \dfrac{\dfrac{K}{s(s+1)(s+2)}}{1+\dfrac{K}{s(s+1)(s+2)}} = \dfrac{K}{s(s+1)(s+2)+K}$

The characteristic equation is, $\quad s(s+1)(s+2)+K=0$

$\therefore s(s^2+3s+2)+K=0 \quad \Rightarrow \quad s^3+3s^2+2s+K=0$

The routh array is constructed as shown below.

The highest power of s in the characteristic polynomial is odd number. Hence form the first r using the coefficients of odd powers of s and form the second row using the coefficients of even pow of s.

s^3	:	1	2
s^2	:	3	K
s^1	:	$\dfrac{6-K}{3}$	
s^0	:	K	

Column-1

$s^1 : \dfrac{3\times 2 - K\times 1}{3}$

$s^1 : \dfrac{6-K}{3}$

$s^0 : \dfrac{\dfrac{6-K}{3}\times K - 0\times}{(6-K)/3}$

$s^0 : K$

For the system to be stable there should not be any sign change in the elements of first colum Hence choose the value of K so that the first column elements are positive.

From s^0 row, for the system to be stable, $K > 0$

From s^1 row, for the system to be stable, $\dfrac{6-K}{3} > 0$

For $\dfrac{6-K}{3} > 0$, the value of K should be less than 6.

$\therefore$ The range of K for the system to be stable is $0<K<6$.

RESULT

The value of K is in the range $0<K<6$ for the system to be stable.

EXAMPLE 5.10

The open loop transfer function of a unity feedback control system is given by,

$G(s) = \dfrac{K}{(s+2)(s+4)(s^2+6s+25)}$.

By applying the routh criterion, discuss the stability of the closed-loop system as a function of K. Determine the value of K which will cause sustained oscillations in the closed-loop system. What are the corresponding oscillating frequencies?

SOLUTION

The closed loop transfer function $\left.\right\}$ $\dfrac{C(s)}{R(s)}=\dfrac{G(s)}{1+G(s)}=\dfrac{\dfrac{K}{(s+2)(s+4)(s^2+6s+25)}}{1+\dfrac{K}{(s+2)(s+4)(s^2+6s+25)}}=\dfrac{K}{(s+2)(s+4)(s^2+6s+25)+K}$

The characteristic equation is given by the denominator polynomial of closed loop transfer function.

The characteristic equation is, $(s+2)(s+4)(s^2+6s+25)+K=0$.

$$\therefore (s^2+6s+8)(s^2+6s+25)+K=0 \quad \Rightarrow \quad s^4+12s^3+69s^2+198s+200+K=0$$

The routh array is constructed as shown below. The highest power of s in the characteristic equation is even number. Hence form the first row using the coefficients of even powers of s and form the second row using the coefficients of odd powers of s.

s^4 : 1 69 200+K Row-1

s^3 : 12 198 ..... Row-2

Divide s^3 row by 12 to simplify the calculations

s^4	:	1	69	200+K	..... Row-1
s^3	:	1	16.5		..... Row-2
s^2	:	52.5	200+K		..... Row-3
s^1	:	$\dfrac{666.25-K}{52.5}$			..... Row-4
s^0	:	200+K			..... Row-5

Column-1

$$s^2 : \frac{1\times 69-16.5\times 1}{1} \quad \frac{1\times(200+K)}{1}$$

$$s^2 : 52.5 \quad 200+K$$

$$s^1 : \frac{52.5\times 16.5-(200+K)\times 1}{52.5}$$

$$s^1 : \frac{666.25-K}{52.5}$$

$$s^0 : \frac{\frac{666.25-K}{52.5}\times(200+K)}{(666.25-K)/52.5}$$

$$s^0 : 200+K$$

For the system to be stable there should not be any sign change in the elements of first column. Hence choose the value of K so that the first column elements are positive.

From s^1 row, for the system to be stable, $(666.25-K)>0$.

Since $(666.25-K)>0$, should be less than 666.25.

From s^0 row, for the system to be stable, $(200+K)>0$

Since $(200+K)>0$, K should be greaterthan -200, but practical values of K starts from 0. Hence K should be greater than 0.

$\therefore$ The range of K for the system to be stable is $0<K<666.25$.

When K= 666.25 the s^1 row becomes zero, which indicates the possibility of roots on imaginary axis. A system will oscillate if it has roots on imaginary axis and no roots on right half of s-plane.

When K=666.25, the coefficients of auxiliary equation are given by the s^2 row.

$\therefore$ The auxiliary equation is, $52.5s^2+200+K=0$

$$52.5s^2+200+666.25=0$$

$$s^2=\frac{-200-666.25}{52.5}=-16.5$$

$$s=\pm\sqrt{-16.5}=\pm j\sqrt{16.5}=\pm j4.06$$

When K = 666.25, the system has roots on imaginary axis and so it oscillates. The frequency of oscillation is given by the value of root on imaginary axis.

$\therefore$ The frequency of oscillation, $\omega = 4.06$ rad/sec.

RESULT

1. The range of K for stability is $0<K<666.25$.
2. The system oscillates when $K = 666.25$
3. The frequency of oscillation, $\omega = 4.06$ rad/sec. (When $K = 666.25$).

EXAMPLE 5.11

The open loop transfer function of a unity feedback system is given by, $G(s)=\frac{K(s+1)}{s^3+as^2+2s+1}$. Determine the value of **K** and **a** so that the system oscillates at a frequency of 2 rad/sec.

SOLUTION

The closed loop transfer function $\left.\right\}$ $\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)}=\frac{\frac{K(s+1)}{s^3+as^2+2s+1}}{1+\frac{K(s+1)}{s^3+as^2+2s+1}}=\frac{K(s+1)}{s^3+as^2+2s+1+K(s+1)}$

The characteristic equation is, $s^3+as^2+2s+1+K(s+1)=0$.

$$s^3+as^2+2s+1+Ks+K=0 \quad \Rightarrow \quad s^3+as^2+(2+K)s+1+K=0$$

The routh array of characteristic polynomial is constructed as shown below. The maximum power of s is odd, hence the first row of routh array is formed using coefficients of odd powers of s and second row of routh array is formed using coefficients of even powers of s.

If the elements of s^1 row are all zeros then there exist an even polynomial (or auxiliary polynomial). If the roots of the auxiliary polynomial are purely imaginary then the roots are lying on imaginary axis and the system oscillates. The frequency of oscillation is the root of auxiliary polynomial.

Routh array

$$\begin{array}{lll}
s^3: & 1 & 2+K \\
s^2: & a & 1+K \\
s^1: & \frac{a(2+K)-(1+K)}{a} & \\
s^0: & 1+K &
\end{array}$$

From s^2 row, the auxiliary polynomial is,

$$as^2+(1+K)=0 \quad \Rightarrow \quad as^2=-(1+K) \quad \Rightarrow \quad s=\pm j\sqrt{\frac{1+K}{a}}$$

Given that, $s=\pm j2$, $\quad \therefore \sqrt{\frac{1+K}{a}}=2 \quad \Rightarrow \quad \frac{1+K}{a}=4 \quad \Rightarrow \quad K=4a-1$

From s^1 row, $\frac{a(2+K)-(1+K)}{a}=0 \Rightarrow a(2+K)-(1+K)=0 \Rightarrow 2a+Ka-1-K=0$

$$\therefore \quad 2a-1+K(a-1)=0$$

Put, $K=4a-1$

$\therefore 2a-1+(4a-1)(a-1)=0 \Rightarrow 2a-1+4a^2-4a-a+1=0 \Rightarrow 4a^2-3a=0$ (or) $a(4a-3)=0$

Since $a \neq 0$, $4a-3=0$, $\therefore a=3/4$

When $a=(3/4)$, $K=4a-1=4\times(3/4)-1=2$

RESULT

When the system oscillates at a frequency of 2 rad/sec, $K=2$ and $a=3/4$.

EXAMPLE 5.12

A feedback system has open loop transfer function of $G(s)=\dfrac{Ke^{-s}}{s(s^2+5s+9)}$. Determine the maximum value of K for stability of closed loop system.

SOLUTION

Generally control systems have very low bandwidth which implies that it has very low frequency range of operation. Hence for low frequency ranges the term e^{-s} can be replaced by, $1-s$, (i.e., $e^{-sT} \approx 1-sT$).

$$\therefore G(s)=\frac{Ke^{-s}}{s(s^2+5s+9)} \approx \frac{K(1-s)}{s(s^2+5s+9)}$$

The closed loop transfer function
$$\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)}=\frac{\dfrac{K(1-s)}{s(s^2+5s+9)}}{1+\dfrac{K(1-s)}{s(s^2+5s+9)}}=\frac{K(1-s)}{s(s^2+5s+9)+K(1-s)}$$

The characteristic equation is given by the denominator polynomial of closed loop transfer function.

$\therefore$ The characteristic equation is, $s(s^2+5s+9)+K(1-s)=0$

$$\therefore s(s^2+5s+9)+K(1-s)=s^3+5s^2+9s+K-Ks=0 \Rightarrow s^3+5s^2+(9-K)s+K=0$$

The routh array of characteristic polynomial is constructed as shown below.

The maximum power of s in the characteristic polynomial is odd, hence form the first row of routh array using coefficients of odd powers of s and second row of routh array using coefficients of even powers of s.

$$\begin{array}{lll} s^3 : & 1 & 9-K \\ s^2 : & 5 & K \\ s^1 : & 9-1.2K & \\ s^0 : & K & \end{array}$$

$$s^1 : \frac{5\times(9-K)-K\times 1}{5}$$
$$s^1 : \frac{45-5K-K}{5}$$
$$s^1 : \frac{45-6K}{5} \approx 9-1.2K$$

$$s^0 : \frac{(9-1.2K)\times K}{(9-1.2K)}$$
$$s^0 : K$$

From s^1 row, for stability of the system, $(9-1.2K)>0$

If $(9-1.2K)>0$ then $1.2K<9$; $\therefore K<\dfrac{9}{1.2}=7.5$

From s^0 row, for stability of the system, $K>0$

Finally we can conclude that for stability of the system K should be in the range of $0<K<7.5$

RESULT

For stability of the system K should be in the range of, $\mathbf{0<K<7.5}$.