N	Denominator of $H(s)$
1	$s+1$
2	$s^2+\sqrt{2}s+1$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.76537s+1)(s^2+1.8477s+1)$
5	$(s+1)(s^2+0.61803s+1)(s^2+1.61803s+1)$
6	$(s^2+1.931855s+1)(s^2+\sqrt{2}s+1)(s^2+0.51764s+1)$
7	$(s+1)(s^2+1.80194s+1)(s^2+1.247s+1)(s^2+0.445s+1)$

Example 5.1 Given the specification $\alpha_p = 1\,\text{dB}$; $\alpha_s = 30\,\text{dB}$; $\Omega_p = 200$ rad/sec; $\Omega_s = 600$ rad/sec. Determine the order of the filter.

Solution

From Eq. (5.25)

$$A = \frac{\lambda}{\varepsilon} = \left(\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}\right)^{0.5}$$

$$= \left(\frac{10^3 - 1}{10^{0.1} - 1}\right)^{0.5} = 62.115$$

From Eq. (5.26)

$$k = \frac{\Omega_p}{\Omega_s} = \frac{200}{600} = \frac{1}{3}$$

From Eq. (5.27)

$$N \geq \frac{\log A}{\log 1/k}$$

$$\geq \frac{\log 62.115}{\log 3} = 3.758$$

Rounding off N to the next higher integer we get $N = 4$.

Example 5.2 Determine the order and the poles of lowpass Butterworth filter that has a 3 dB attenuation at 500 Hz and an attenuation of 40 dB at 1000 Hz.

Solution

Given data $\alpha_p = 3\,\text{dB}$; $\alpha_s = 40\,\text{dB}$; $\Omega_p = 2 \times \pi \times 500 = 1000\pi$ rad/sec. $\Omega_s = 2 \times \pi \times 1000 = 2000\pi$ rad/sec.

order of the filter

$$N \geq \frac{\log \sqrt{\dfrac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\log \dfrac{\Omega_s}{\Omega_p}}$$

$$\geq \frac{\log \sqrt{\dfrac{10^4 - 1}{10^{0.3} - 1}}}{\log \dfrac{2000\pi}{1000\pi}} = 6.6$$

Example 5.4 Design an analog Butterworth filter that has a $-2\,\text{dB}$ passband attenuation at a frequency of 20 rad/sec and atleast $-10\,\text{dB}$ stopband attenuation at 30 rad/sec.

Solution

Given $\alpha_p = 2\,\text{dB}; \quad \Omega_p = 20\,\text{rad/sec}$

$\alpha_s = 10\,\text{dB}; \quad \Omega_s = 30\,\text{rad/sec}$

$$N \geq \frac{\log\sqrt{\dfrac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\dfrac{\Omega_s}{\Omega_p}}$$

$$\geq \frac{\log \sqrt{\frac{10-1}{10^{0.2}-1}}}{\log \frac{30}{20}}$$

$$\geq 3.37$$

Rounding off N to the next highest integer we get

$$N = 4$$

The normalized lowpass Butterworth filter for $N = 4$ can be found from table 5.1 as

$$H(s) = \frac{1}{(s^2 + 0.76537s + 1)(s^2 + 1.8477s + 1)}$$

From Eq. (5.31) we have

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p} - 1)^{1/2N}} = \frac{20}{(10^{0.2} - 1)^{1/8}} = 21.3868$$

> **Note:**
> To find the cutoff frequency Ω_c either (5.31) or (5.32a) can be used. The Eq. (5.31) satisfies passband specification at Ω_p, while the stopband specification at Ω_s is exceeded. The Eq. (5.32a) satisfies the stopband specification Ω_s, while the passband specification at Ω_p is exceeded. All the examples in this chapter are solved using Eq. (5.31). Students are advised to solve the exercise problems using Eq. (5.32a).

The transfer function for $\Omega_c = 21.3868$ can be obtained by substituting

$$s \to \frac{s}{21.3868} \text{ in } H(s)$$

i.e.,

$$H(s) = \frac{1}{\left(\frac{s}{21.3868}\right)^2 + 0.76537\left(\frac{s}{21.3868}\right) + 1} \times \frac{1}{\left(\frac{s}{21.3868}\right)^2 + 1.8477\left(\frac{s}{21.3868}\right) + 1}$$

$$= \frac{0.20921 \times 10^6}{(s^2 + 16.3686s + 457.394)(s^2 + 39.5176s + 457.394)};$$

Example 5.5 For the given specifications design an analog Butterworth filter. $0.9 \leq |H(j\Omega)| \leq 1$ for $0 \leq \Omega \leq 0.2\pi$. $|H(j\Omega)| \leq 0.2$ for $0.4\pi \leq \Omega \leq \pi$.

Solution

From the data we find $\Omega_p = 0.2\pi$; $\Omega_s = 0.4\pi$; $\frac{1}{\sqrt{1+\varepsilon^2}} = 0.9$ and $\frac{1}{\sqrt{1+\lambda^2}} = 0.2$ from which we obtain

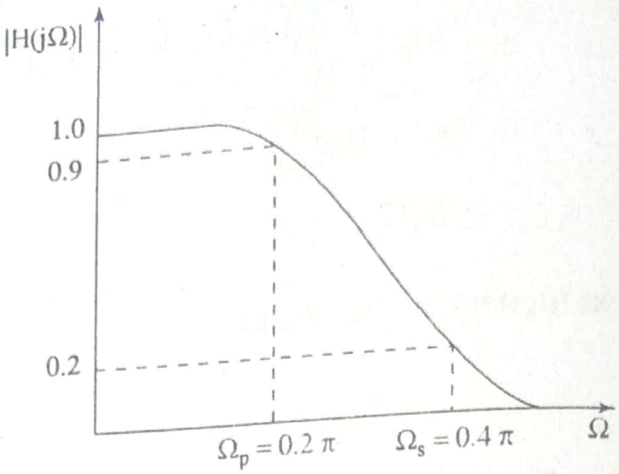


Fig. 5.8 Magnitude response of example 5.5

$$\varepsilon = 0.484 \text{ and } \lambda = 4.898$$

$$N \geq \frac{\log\left(\frac{\lambda}{\varepsilon}\right)}{\log\frac{\Omega_s}{\Omega_p}} = \frac{\log\frac{4.898}{0.484}}{\log\left(\frac{0.4\pi}{0.2\pi}\right)} = 3.34$$

i.e., $N = 4$

From the table 5.1, for $N = 4$, the transfer function of normalised Butterworth filter is

$$H(s) = \frac{1}{(s^2 + 0.76537s + 1)(s^2 + 1.8477s + 1)}$$

we know $\Omega_c = \dfrac{\Omega_p}{(10^{0.1\alpha_p} - 1)^{1/2N}} = \dfrac{\Omega_p}{\varepsilon^{1/N}} = \dfrac{0.2\pi}{(0.484)^{1/4}} = 0.24\pi.$

$H(s)$ for $\Omega_c = 0.24\pi$ can be obtained by substituting $s \to \dfrac{s}{0.24\pi}$ in $H(s)$ i.e.,

$$H(s) = \frac{1}{\left\{\left(\frac{s}{0.24\pi}\right)^2 + 0.76537\left(\frac{s}{0.24\pi}\right) + 1\right\}}$$

$$\times \frac{1}{\left(\frac{s}{0.24\pi}\right)^2 + 1.8477\left(\frac{s}{0.24\pi}\right) + 1}$$

$$= \frac{0.323}{(s^2 + 0.577s + 0.0576\pi^2)(s^2 + 1.393s + 0.0576\pi^2)}$$

Practice Problem 5.1 For the given specifications find the order of the Butterworth filter

$$\alpha_p = 3\,\text{dB}; \quad \alpha_s = 18\,\text{dB}; \quad f_p = 1\,\text{kHz}; \quad f_s = 2\,\text{kHz}.$$

Practice Problem 5.2 Design an analog Butterworth filter that has

$$\alpha_p = 0.5\,\text{dB}; \quad \alpha_s = 22\,\text{dB}; \quad f_p = 10\,\text{kHz}; \quad f_s = 25\,\text{kHz}.$$

Example 5.6 Given the specifications $\alpha_p = 3\,\text{dB}$; $\alpha_s = 16\,\text{dB}$; $f_p = 1\,\text{KHz}$ and $f_s = 2\,\text{KHz}$. Determine the order of the filter using Chebyshev approximation. Find $H(s)$.

Solution

From the given data we can find

$$\Omega_p = 2\pi \times 1000\,\text{Hz} = 2000\,\pi\ \text{rad/sec}$$
$$\Omega_s = 2\pi \times 2000\,\text{Hz} = 4000\,\pi\ \text{rad/sec}$$

and $\alpha_p = 3\,\text{dB}$; $\alpha_s = 16\,\text{dB}$.

Step 1:

$$N \geq \frac{\cosh^{-1}\sqrt{\dfrac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\cosh^{-1}\dfrac{\Omega_s}{\Omega_p}} = \frac{\cosh^{-1}\sqrt{\dfrac{10^{1.6}-1}{10^{0.3}-1}}}{\cosh^{-1}\dfrac{4000\pi}{2000\pi}}$$
$$= 1.91$$

Step 2: Rounding N to next higher value we get $N = 2$.

For N even, the oscillatory curve starts from $\dfrac{1}{\sqrt{1+\varepsilon^2}}$.

Step 3: The values of minor axis and major axis can be found as below

$$\varepsilon = (10^{0.1\alpha_p} - 1)^{0.5} = (10^{0.3} - 1)^{0.5} = 1$$
$$\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}} = 2.414$$
$$a = \Omega_p \frac{[\mu^{1/N} - \mu^{-1/N}]}{2} = 2000\pi \frac{[(2.414)^{1/2} - (2.414)^{-1/2}]}{2} = 910\pi$$
$$b = \Omega_p \frac{[\mu^{1/N} + \mu^{-1/N}]}{2} = 2000\pi \frac{[(2.414)^{1/2} + (2.414)^{-1/2}]}{2} = 2197\pi$$

$$s_k = a\cos\phi_k + jb\sin\phi_k, \quad k = 1,2$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \quad k = 1,2$$

$$\phi_1 = \frac{\pi}{2} + \frac{\pi}{4} = 135^\circ$$

$$\phi_2 = \frac{\pi}{2} + \frac{3\pi}{4} = 225^\circ$$

$$s_1 = a\cos\phi_1 + j\,b\sin\phi_1 = -643.46\pi + j1554\pi$$

$$s_2 = a\cos\phi_2 + j\,b\sin\phi_2 = -643.46\pi - j1554\pi$$

Step 5: The denominator of $H(s) = (s + 643.46\pi)^2 + (1554\pi)^2$

Step 6: The numerator of $H(s) = \dfrac{(643.46\pi)^2 + (1554\pi)^2}{\sqrt{1+\varepsilon^2}} = (1414.38)^2\pi^2$

The transfer function $H(s) = \dfrac{(1414.38)^2\pi^2}{s^2 + 1287\pi s + (1682)^2\pi^2}$.

Example 5.7 Obtain an analog Chebyshev filter transfer function that satisfies the constraints $\frac{1}{\sqrt{2}} \le |H(j\Omega)| \le 1; \quad 0 \le \Omega \le 2$

$$|H(j\Omega)| < 0.1; \quad \Omega \ge 4$$

Solution

Step 1: From the given data we can find that

$$\frac{1}{\sqrt{1+\varepsilon^2}} = \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{1+\lambda^2}} = 0.1,$$

$\Omega_p = 2$ and $\Omega_s = 4$, from which we can obtain $\varepsilon = 1$ and $\lambda = 9.95$.

We know

$$N \ge \frac{\cosh^{-1}\frac{\lambda}{\varepsilon}}{\cosh^{-1}\frac{\Omega_s}{\Omega_p}} = \frac{\cosh^{-1}9.95}{\cosh^{-1}2} = 2.269$$

Step 2: Rounding N to next higher value we get $N = 3$. For N odd, the oscillatory curve starts from unity as shown in Fig. 5.12.

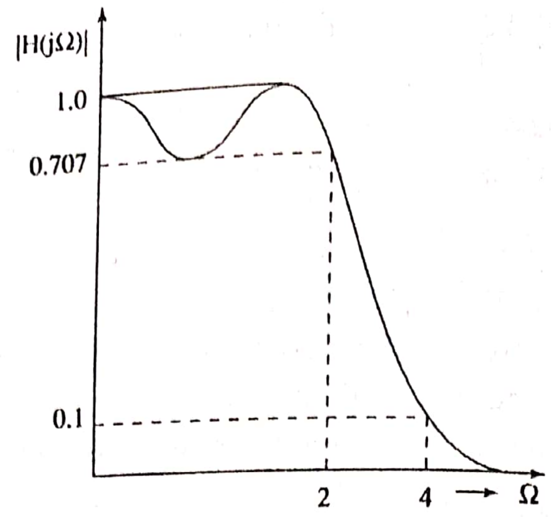


Fig. 5.12 Magnitude response of example 5.7.

Step 3: Finding the values of a and b

$$\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}} = 2.414$$

$$a = \Omega_p \left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right] = 2\left[\frac{(2.414)^{1/3} - (2.414)^{-1/3}}{2}\right]$$

$$= 0.596$$

$$b = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right] = 2\left[\frac{(2.414)^{1/3} + (2.414)^{-1/3}}{2}\right]$$

$$= 2.087$$

Step 4: To calculate the poles of Chebyshev filter

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \quad k = 1, 2, 3$$

$$\phi_1 = 120°,\ \phi_2 = 180°,\ \phi_3 = 240°$$

We know $s_k = a\cos\phi_k + jb\sin\phi_k \quad k = 1, 2, 3$ from which we get

$$s_1 = a\cos\phi_1 + jb\sin\phi_1 = 0.596\cos 120° + j2.087\sin 120° = -0.298 + j1.807$$

$$s_2 = a\cos\phi_2 + jb\sin\phi_2 = 0.596\cos 180° + j2.087\sin 180° = -0.596$$

$$s = a\cos\phi_3 + jb\sin\phi_3 = 0.596\cos 240° + j2.087\sin 240° = -0.298 - j1.807$$

Step 5: The denominator polynomial is given by

$$(s+0.596)\{(s+0.298) - j1.807\}\{(s+0.298) + j1.807\}$$

$$(s+0.596)[(s+0.298)^2 + (1.807)^2]$$

$$= (s+0.596)(s^2 + 0.596s + 3.354)$$

Step 6: The numerator of $H(s)$ can be obtained by substituting $s = 0$ (for N odd) in the denominator.

Therefore the numerator of $H(s) = 2$

The transfer function of Chebyshev filter for the given specifications is given by

$$H(s) = \frac{2}{(s + 0.596)(s^2 + 0.596s + 3.354)}$$

Example 5.8 Determine the order and the poles of a type I lowpass Chebyshev filter that has a 1 dB ripple in the passband and passband frequency $\Omega_p = 1000\pi$, a stopband frequency of 2000π and an attenuation of 40 dB or more.

Solution

Given data $\alpha_p = 1$ dB; $\Omega_p = 1000\pi$ rad/sec; $\alpha_s = 40$ dB

$\Omega_s = 2000\pi$ rad/sec

$$N \geq \frac{\cos h^{-1}\sqrt{\dfrac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\cos h^{-1}\dfrac{\Omega_s}{\Omega_p}} \geq \frac{\cos h^{-1}\sqrt{\dfrac{10^4 - 1}{10^{0.1} - 1}}}{\cos h^{-1}\dfrac{2000\pi}{1000\pi}} = 4.536$$

i.e., $N = 5$

$$\varepsilon = \sqrt{10^{0.1\alpha_p} - 1} = 0.508; \quad \mu = \varepsilon^{-1} + \sqrt{1 + \varepsilon^{-2}} = 4.17$$

$$a = \Omega_p\left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right] = 289.5\pi; \quad b = \Omega_p\left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right] = 1041\pi$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \quad k = 1, 2, \ldots 5$$

$\phi_1 = 180°$; $\phi_2 = 144°$; $\phi_3 = 180°$; $\phi_4 = 216°$; $\phi_5 = 252°$

$s_k = a\cos\phi_k + jb\sin\phi_k \quad k = 1, 2, \ldots 5$

$s_1 = -89.5\pi + j989\pi$; $\quad s_2 = -234.2\pi + j612\pi$; $\quad s_3 = -289.5\pi$

$s_4 = -234.2\pi - j612\pi$; $\quad s_5 = -89.5\pi - j989\pi$

Example 5.9 Design a Chebyshev filter with a maximum passband attenuation of 2.5 dB; at $\Omega_p = 20$ rad/sec and the stopband attenuation of 30 dB at $\Omega_s = 50$ rad/sec.

(AU ECE May'07)

Solution

Given

$\Omega_p = 20$ rad/sec; $\quad \alpha_p = 2.5$ dB;

$\Omega_s = 50$ rad/sec; $\quad \alpha_s = 30$ dB;

We know

$$N = \frac{\cosh^{-1} \lambda/\varepsilon}{\cosh^{-1} 1/k}$$

$$\lambda = \sqrt{10^{0.1\alpha_s} - 1} = 31.607$$

$$\varepsilon = \sqrt{10^{0.1\alpha_p} - 1} = 0.882$$

$$k = \frac{\Omega_p}{\Omega_s} = 0.4$$

Now

$$N \geq \frac{\cosh^{-1} \dfrac{31.607}{0.882}}{\cosh^{-1} \dfrac{1}{0.4}} = 2.726$$

i.e., $N = 3$

$$\mu = \varepsilon^{-1} + \sqrt{1 + \varepsilon^{-2}} = 2.65$$

$$a = \Omega_p \frac{[\mu^{1/N} - \mu^{-1/N}]}{2} = 6.6$$

$$b = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2} \right] = 21.06$$

$$s_k = a \cos \phi_k + jb \sin \phi_k; \quad k = 1, 2, 3$$

$$\phi_k = \frac{\pi}{2} + \left(\frac{2k - 1}{2N} \right) \pi; \quad k = 1, 2, 3$$

$$\phi_1 = 120°, \phi_2 = 180°, \phi_3 = 240°$$

$$s_1 = -3.3 + j18.23$$

$$s_2 = -6.6$$

$$s_3 = -3.3 - j18.23$$

Denominator of $H(s) = (s + 6.6)(s^2 + 6.6s + 343.2)$

Numerator of $H(s) = (6.6)(343.2) = 2265.27$

Transfer function $H(s) = \dfrac{2265.27}{(s + 6.6)(s^2 + 6.6s + 343.2)}$

Practice Problem 5.4 For the ... Chebyshev-I ods described in sections 5.8 and 5.9 and then use suitable transformation

Example 5.10 For the given specifications $\alpha_p = 3$ dB; $\alpha_s = 15$ dB; $\Omega_p = 1000$ rad/sec and $\Omega_s = 500$ rad/sec design a highpass filter.

Solution

First we design a normalized lowpass filter and then use suitable transformation to get the transfer function of a highpass filter.

For lowpass filter	For highpass filter
$\Omega_c = \Omega_p = 500$ rad/sec	$\Omega_c = \Omega_p = 1000$ rad/sec
$\Omega_s = 1000$ rad/sec	$\Omega_s = 500$ rad/sec

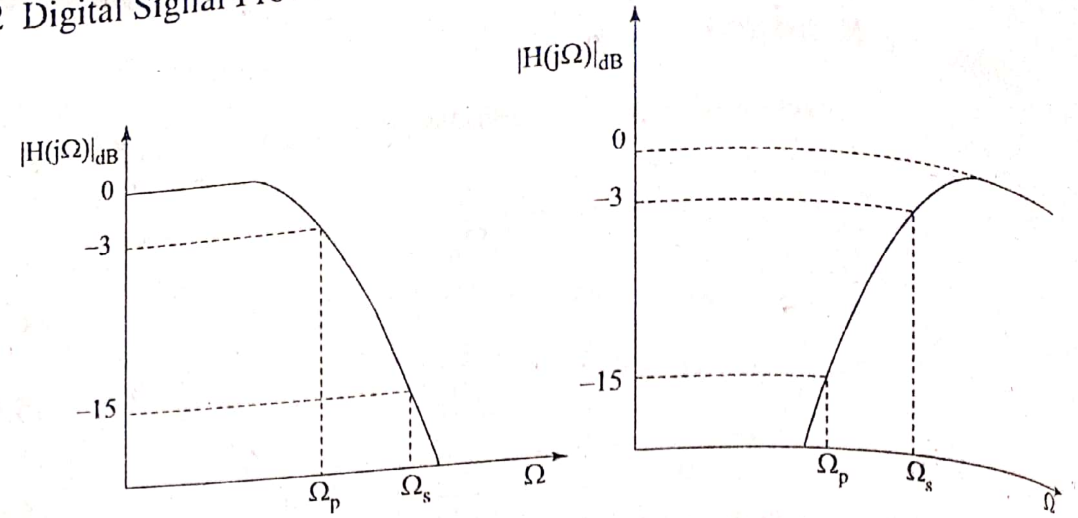


Fig. 5.17 Lowpass to highpass transformation

Lowpass filter specifications

$$\Omega_c = \Omega_p = 500\,\text{rad/sec}; \quad \alpha_p = 3\,\text{dB}$$
$$\Omega_s = 1000\,\text{rad/sec}; \quad \alpha_s = 15\,\text{dB}$$

We have

$$N = \frac{\log \dfrac{\lambda}{\varepsilon}}{\log 1/k}$$
$$\lambda = \sqrt{10^{0.1\alpha_s} - 1} = 5.533$$
$$\varepsilon = \sqrt{10^{0.1\alpha_p} - 1} = 1$$
$$k = \frac{\Omega_p}{\Omega_s} = 0.5$$

Therefore $N = \frac{\log 5.533}{\log 2} = 2.468$. Approximating to next higher integer we have $N = 3$.

$H(s)$ for $\Omega_c = 1$ rad/sec and $N = 3$ is

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

To get highpass filter having cutoff frequency

$$\Omega_c = \Omega_p = 1000\,\text{rad/sec}$$

Substitute $s \to \dfrac{1000}{s}$

$$H_a(s) = H(s)\Big|_{s \to \frac{1000}{s}}$$
$$= \frac{1}{(s+1)(s^2+s+1)}\Big|_{s \to \frac{1000}{s}}$$
$$= \frac{s^3}{(s+1000)[s^2+1000s+(1000)^2]}$$

Example 5.11 For the analog transfer function $H(s) = \frac{2}{(s+1)(s+2)}$ determine $H(z)$ using impulse invariance method. Assume $T = 1$ sec.

Solution

Given $H(s) = \dfrac{2}{(s+1)(s+2)}$

Using partial fraction we can write

$$A = (s+1)\frac{2}{(s+1)(s+2)}\bigg|_{s=-1} = 2$$

$$B = (s+2)\frac{2}{(s+1)(s+2)}\bigg|_{s=-2} = -2$$

$$H(s) = \frac{A}{s+1} + \frac{B}{s+2}$$

$$H(s) = \frac{2}{s+1} - \frac{2}{s+2}$$

$$= \frac{2}{s-(-1)} - \frac{2}{s-(-2)}$$

Using impulse invariance technique we have, if

$$H(s) = \sum_{k=1}^{N} \frac{c_k}{s-p_k} \quad \text{then} \quad H(z) = \sum_{k=1}^{N} \frac{c_k}{1-e^{p_kT}z^{-1}}$$

i.e., $(s - p_k)$ is transformed to $1 - e^{p_kT}z^{-1}$.

There are two poles $p_1 = -1$ and $p_2 = -2$. So

$$H(z) = \frac{2}{1-e^{-T}z^{-1}} - \frac{2}{1-e^{-2T}z^{-1}}$$

For $T = 1$ sec

$$H(z) = \frac{2}{1-e^{-1}z^{-1}} - \frac{2}{1-e^{-2}z^{-1}}$$

$$= \frac{2}{1-0.3678z^{-1}} - \frac{2}{1-0.1353z^{-1}}$$

$$H(z) = \frac{0.465z^{-1}}{1-0.503z^{-1}+0.04976z^{-2}}$$

Example 5.13 Design a third order Butterworth digital filter using impulse invariant technique. Assume sampling period $T = 1$ sec.

Solution

From the table 5.1, for $N = 3$, the transfer function of a normalised Butterworth filter is given by

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$
$$= \frac{A}{s+1} + \frac{B}{s+0.5+j0.866} + \frac{C}{s+0.5-j0.866}$$

$$A = (s+1)\frac{1}{(s+1)(s^2+s+1)}\bigg|_{s=-1} = \frac{1}{(-1)^2-1+1} = 1$$

$$B = (s+0.5+j0.866)\frac{1}{(s+1)(s+0.5+j0.866)(s+0.5-j0.866)}\bigg|_{s=-0.5-j0.866}$$

$$= \frac{1}{(-0.5-j0.866+1)(-j0.866-j0.866)}$$

$$= \frac{1}{-j1.732(0.5-j0.866)} = \frac{1}{-j0.866-1.5}$$

$$= \frac{-1.5+j0.866}{3} = -0.5+j0.288$$

$$C = B^* = -0.5-j0.288$$

Hence

$$H(s) = \frac{1}{s+1} + \frac{-0.5+0.288j}{s+0.5+j0.866} + \frac{-0.5-0.288j}{s+0.5-j0.866}$$

$$= \frac{1}{s-(-1)} + \frac{-0.5+0.288j}{s-(-0.5-j0.866)} + \frac{-0.5-0.288j}{s-(-0.5+j0.866)}$$

In impulse invariant technique

$$\text{if} \quad H(s) = \sum_{k=1}^{N} \frac{c_k}{s-p_k}, \quad \text{then} \quad H(z) = \sum_{k=1}^{N} \frac{c_k}{1-e^{p_kT}z^{-1}}$$

Therefore,

$$H(z) = \frac{1}{1-e^{-1}z^{-1}} + \frac{-0.5+j0.288}{1-e^{-0.5}e^{-j0.866}z^{-1}} + \frac{-0.5-j0.288}{1-e^{-0.5}e^{j0.866}z^{-1}}$$

$$= \frac{1}{1-0.368z^{-1}} + \frac{-1+0.66z^{-1}}{1-0.786z^{-1}+0.368z^{-2}}$$

Example 5.16 Apply bilinear transformation to $H(s) = \frac{2}{(s+1)(s+2)}$ with $T = 1$ sec and find $H(z)$.

Solution

Given $H(s) = \frac{2}{(s+1)(s+2)}$

Substitute $s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$ in $H(s)$ to get $H(z)$

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{2}{(s+1)(s+2)}\Bigg|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

Given $T = 1$ sec

$$H(z) = \frac{2}{\left\{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+1\right\}\left\{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+2\right\}}$$

$$= \frac{2(1+z^{-1})^2}{(3-z^{-1})(4)}$$

$$= \frac{(1+z^{-1})^2}{6-2z^{-1}}$$

$$= \frac{0.166(1+z^{-1})^2}{(1-0.33z^{-1})}$$

Example 5.17 Using the bilinear transform, design a highpass filter, monotonic in passband with cutoff frequency of 1000 Hz and down 10 dB at 350 Hz. The sampling frequency is 5000 Hz.

Solution

Given $\alpha_p = 3\,\text{dB}; \omega_c = \omega_p = 2 \times \pi \times 1000 = 2000\pi$ rad/sec

$$\alpha_s = 10\,\text{dB}; \quad \omega_s = 2 \times \pi \times 350 = 700\pi \text{ rad/sec}$$

$$T = \frac{1}{f} = \frac{1}{5000} = 2 \times 10^{-4} sec$$

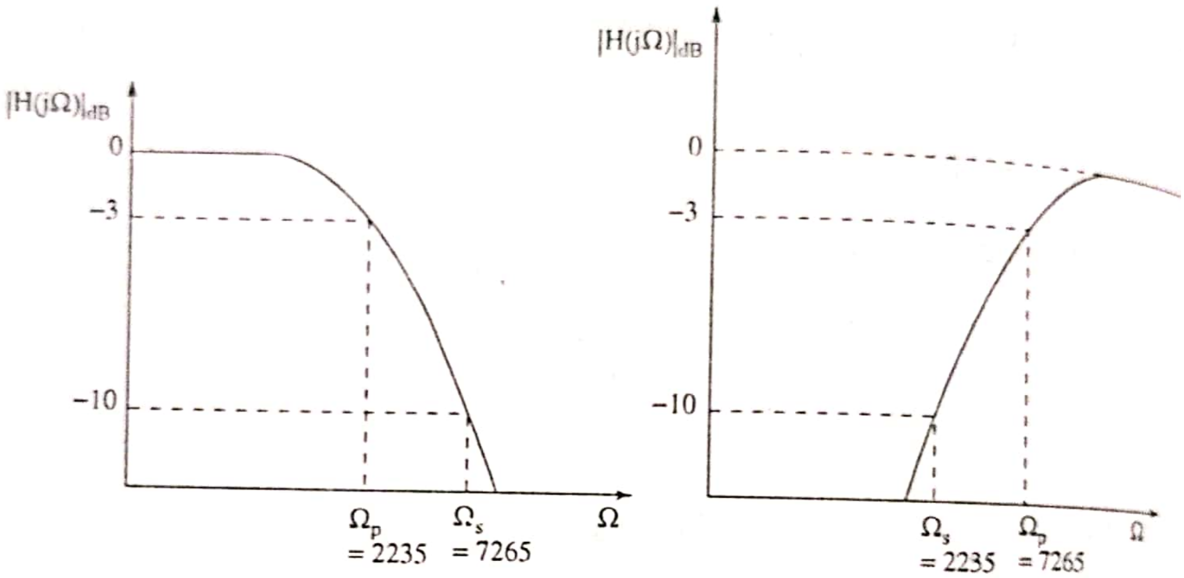


Fig. 5.27

The characteristics are monotonic in both passband and stopband. Therefore, the filter is Butterworth filter.

Prewarping the digital frequencies we have

$$\Omega_p = \frac{2}{T}\tan\frac{\omega_p T}{2} = \frac{2}{2\times 10^{-4}}\tan\frac{(2000\pi \times 2\times 10^{-4})}{2}$$
$$= 10^4\tan(0.2\pi) = 7265\,\text{rad/sec}$$

$$\Omega_s = \frac{2}{T}\tan\frac{\omega_s T}{2} = \frac{2}{2\times 10^{-4}}\tan\frac{(700\pi \times 2\times 10^{-4})}{2}$$
$$= 10^4\tan(0.07\pi) = 2235\text{rad/sec}$$

First we design a lowpass filter for the given specifications and use suitable transformation to obtain transfer function of highpass filter.

The order of the filter

$$N = \frac{\log\sqrt{\dfrac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\dfrac{\Omega_s}{\Omega_p}} = \frac{\log\sqrt{\dfrac{10^{0.1(10)}-1}{10^{0.1(3)}-1}}}{\log\dfrac{7265}{2235}} = \frac{\log 3}{\log 3.25} = \frac{0.4771}{0.5118} = 0.932$$

Therefore, we take $N = 1$.

The first-order Butterworth filter for $\Omega_c = 1$ rad/sec is $H(s) = \dfrac{1}{1+s}$

The highpass filter for $\Omega_c = \Omega_p = 7265$ rad/sec can be obtained by using the transformation

$$s \rightarrow \frac{\Omega_c}{s}$$

$$i.e., \; s \rightarrow \frac{(7265)}{s}$$

The transfer function of highpass filter

$$H(s) = \frac{1}{s+1}\bigg|_{s=\frac{7265}{s}}$$

$$= \frac{s}{s+7265}$$

Using bilinear transformation

$$H(z) = H(s)\bigg|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{s}{s+7265}\bigg|_{s=\frac{2}{2\times10^{-4}}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{10000\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}{10000\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+7265}$$

$$= \frac{0.5792(1-z^{-1})}{1-0.1584z^{-1}}$$

lowpass filter.

Butterworth

Example 5.33 Determine the system function $H(z)$ of the lowest order Chebyshev and Butterworth digital filter with the following specification

(a) $3db$ ripple in pass band $0 \leq \omega \leq 0.2\pi$

(b) $25db$ attenuation in stop band $0.45\pi \leq \omega \leq \pi$

JNTU Nov'05 (set 3)

Butterworth filter

Using bilinear transformation

$$\omega_p = 0.2\pi; \quad \omega_s = 0.45\pi; \quad \alpha_p = 3db; \quad \alpha_s = 25db; \quad T = 1$$

$$\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2} = \frac{2}{T}\tan(0.1\pi) = 0.65$$

$$\Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2} = \frac{2}{T}\tan\left(\frac{0.45\pi}{2}\right) = 1.71$$

$$N \geq \frac{\log\sqrt{\dfrac{10^{2.5}-1}{10^{0.3}-1}}}{\log\dfrac{\Omega_s}{\Omega_p}} = 2.97$$

$$N = 3$$

$$\Omega_p = \Omega_c = 0.65$$

For $N = 3$

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

$$H_a(s) = H(s)\Big|_{s\to\frac{s}{0.65}} = \frac{(0.65)^3}{(s+0.65)(s^2+0.65s+0.4225)}$$

$$H(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{(0.65)^3(1+z^{-1})^3}{[2(1-z^{-1})+0.65(1+z^{-1})]\,[4(1-z^{-1})^2+0.65(1-z^{-2})+0.4225(1+z^{-1})^2]}$$

$$= \frac{(0.65)^3(1+z^{-1})^3}{(2.65-1.35z^{-1})\,(4-8z^{-1}+4z^{-2}+0.65-0.65z^{-2}+0.4225+0.4225z^{-2}+0.845z^{-1})}$$

$$= \frac{(0.65)^3(1+z^{-1})^3}{(2.65-1.35z^{-1})(5.0725-7.155z^{-1}+3.7725z^{-2})}$$
$$= \frac{0.02066(1+z^{-1})^3}{(1-0.51z^{-1})(1-1.41z^{-1}+0.751z^{-2})}$$

Chebyshev filter

$$N \geq \frac{\cosh^{-1}\sqrt{\frac{10^{2.5}-1}{10^{0.3}-1}}}{\cos h^{-1}\frac{\Omega_s}{\Omega_p}}$$

$$\Rightarrow \quad N = 3$$

$$\epsilon = \sqrt{10^{0.3}-1} = 1$$
$$\mu = \epsilon^{-1} + \sqrt{1+\epsilon^{-2}} = 2.414$$
$$a = \Omega_p\left[\frac{\mu^{1/N}-\mu^{-1/N}}{2}\right] = 0.65\left[\frac{(2.414)^{1/3}-(2.414)^{-1/3}}{2}\right] = 0.1935$$
$$b = \Omega_p\left[\frac{\mu^{1/N}+\mu^{-1/N}}{2}\right] = 0.65\left[\frac{(2.414)^{1/3}+(2.414)^{-1/3}}{2}\right] = 0.678$$
$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \qquad k = 1,2,3$$

$$\phi_1 = 120°; \quad \phi_2 = 180°; \quad \phi_3 = 240°$$

$$\begin{aligned} s_1 &= a\cos\phi_1 + jb\sin\phi_1 \\ &= 0.1935\cos(120°) + j0.678\sin(120°) \\ &= -0.09675 + j0.587 \\ s_2 &= a\cos\phi_2 + jb\sin\phi_2 \\ &= 0.1935\cos(180°) + j0.678\sin(180°) \\ &= -0.1935 \\ s_3 &= a\cos\phi_3 + jb\sin\phi_3 \\ &= 0.1935\cos(240°) + j0.678\sin(240°) \\ &= -0.09675 - j0.587 \end{aligned}$$

The denominator polynomial of $H(s) = (s+0.1935)\left[(s+0.09675)^2+0.587^2\right]$
$= (s+0.1935)\left[(s^2+0.1935s+0.354)\right]$

The transfer of $H(s) = (0.1935)(0.354) = 0.0685$

The transfer function $H(s) = \dfrac{0.0685}{(s+0.1935)(s^2+0.1935s+0.354)}$

$$H(z) = H(s)\Big|_{s=\frac{2(1-z^{-1})}{1+z^{-1}}}$$

$$= \frac{0.0685(1+z^{-1})^3}{(2.1935-1.8065z^{-1})(4.5475-7.292z^{-1}+4.1605z^{-2})}$$

$$= \frac{0.00687(1+z^{-1})^3}{(1-0.823z^{-1})(1-1.6z^{-1}+0.915z^{-2})}.$$

Example 5.34 Design a Chebyshev filter for the following specification using (a) bilinear transformation (b) impulse invariance Method.

$$0.8 \leq |H(e^{j\omega})| \leq 1 \qquad 0 \leq \omega \leq 0.2\pi$$
$$|H(e^{j\omega})| \leq 0.2 \qquad 0.6\pi \leq \omega \leq \pi$$

Solution

(a) Given $\omega_s = 0.6\pi$, $\omega_p = 0.2\pi$

$$\frac{1}{\sqrt{1+\lambda^2}} = 0.2 \Rightarrow \lambda = 4.899$$

$$\frac{1}{\sqrt{1+\varepsilon^2}} = 0.8 \Rightarrow \varepsilon = 0.75$$

$$\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2} = 0.6498 \quad (\because T = 1 \text{ sec})$$

$$\Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2} = 2.752$$

$$N = \frac{\cosh^{-1}\lambda/\varepsilon}{\cosh^{-1}1/k} = 1.208$$

$$\Rightarrow N = 2$$

$$\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}} = 3$$

$$a = \Omega_p\left[\frac{\mu^{1/N}-\mu^{-1/N}}{2}\right] = 0.3752$$

$$b = \Omega_p\left[\frac{\mu^{1/N}+\mu^{-1/N}}{2}\right] = 0.75$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \qquad k = 1, 2$$

$$s_k = a\cos\phi_k + jb\sin\phi_k$$

$$s_1 = -0.2653 + j0.53$$

Denominator of

$$H(s) = (s + 0.2653)^2 + (0.53)^2$$
$$= s^2 + 0.5306s + 0.3516$$

For N even, Numerator of $H(s)$ is $\dfrac{0.3516}{[1 + (0.75)^2]^{1/2}} = 0.28$

$$H(s) = \frac{0.28}{s^2 + 0.5306s + 0.3516}$$

Using bilinear transformation

$$H(z) = H(z)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} \qquad \boxed{\because T = 1 \text{ sec}}$$

$$H(z) = \frac{0.28(1 + z^{-1})^2}{5.4128 - 7.298z^{-1} + 3.29z^{-2}}$$
$$= \frac{0.052(1 + z^{-1})^2}{1 - 1.3480z^{-1} + 0.608z^{-2}}$$

(b) By using impulse invariance method

$$\omega = \Omega T \Rightarrow \omega_p = \Omega_p T \quad \text{and} \quad \omega_s = \Omega_s T$$

For $T = 1$ sec

$$\frac{\omega_s}{\omega_p} = \frac{\Omega_s}{\Omega_p} = \frac{0.6\pi}{0.2\pi} = 3$$

$$N = \frac{\cosh^{-1}\frac{\lambda}{\varepsilon}}{\cosh^{-1}\frac{1}{k}} = \frac{\cosh^{-1}\frac{4.899}{0.75}}{\cosh^{-1} 3} = 1.45$$

Approximating N to next higher integer, we get $N = 2$. We know $\mu = 3$

$$a = \Omega_p \left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right] = 0.3627$$

$$b = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right] = 0.7255$$

$$\phi_1 = 135^\circ; \phi_2 = 225^\circ$$
$$s_1 = -0.2564 + j0.513$$
$$s_2 = -0.2564 - j0.513$$

Numerator of $H(s) = 0.264$

$$H(s) = \frac{0.264}{s^2 + 0.513s + 0.33} = \frac{(0.5146)(0.513)}{(s + 0.2564)^2 + (0.513)^2}$$

Example 5.35 Design a bandstop Butterworth and Chebyshev type-I filter to meet the following specifications

(a) Stopband 100 to 600 Hz.

(b) 20 dB attenuation at 200 and 400 Hz.

(c) The gain at $\omega = 0$ is unity.

(d) The passband ripple for the chebyshev filter is 1.1 dB.

(e) The passband attenuation for Butterworth filter is 3 dB.

Solution

Given

$f_l = 100$ Hz; $f_1 = 200$ Hz; $f_2 = 400$ Hz; $f_u = 600$; Hz

Then

$$\Omega_l = 2 \times \pi \times 100 = 200\pi \quad rad/sec$$
$$\Omega_1 = 2 \times \pi \times 200 = 400\pi \quad rad/sec$$
$$\Omega_2 = 2 \times \pi \times 400 = 800\pi \quad rad/sec$$
$$\Omega_u = 2 \times \pi \times 600 = 1200\pi \quad rad/sec$$

First we design a prototype normalized lowpass filter and then use suitable transformation to obtain the transfer function of bandstop filter.

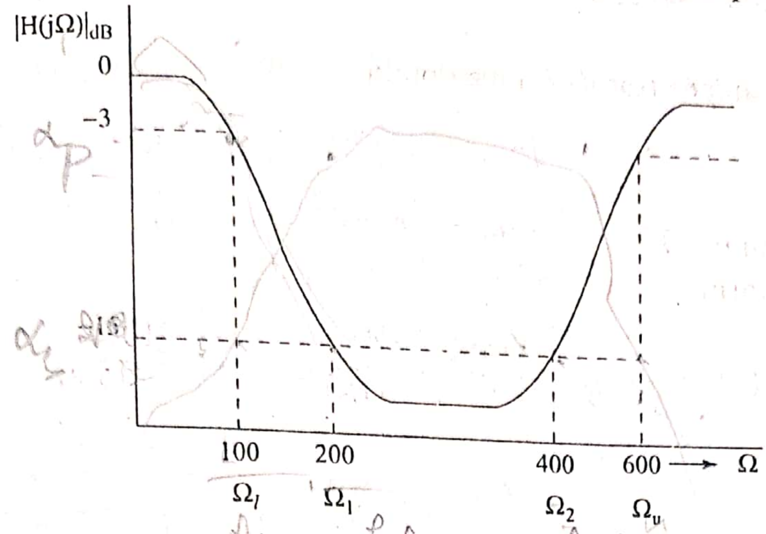


Fig. 5.64

For the normalized lowpass filter

$$\Omega_r = \min\{|A|, |B|\}$$

where

$$A = \frac{\Omega_1(\Omega_u - \Omega_l)}{-\Omega_1^2 + \Omega_l\Omega_u} = \frac{400\pi[1200\pi - 200\pi]}{-(400\pi)^2 + (200\pi)(1200\pi)}$$

$$= 5$$

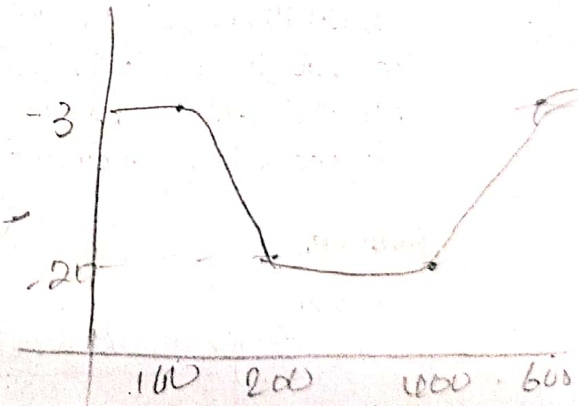
$$B = \frac{\Omega_2(\Omega_l - \Omega_u)}{-\Omega_2^2 + \Omega_l\Omega_u} = \frac{800\pi[1200\pi - 200\pi]}{-(800\pi)^2 + (200\pi)(1200\pi)}$$

$$= -2$$

$\Omega_r = \min\ \{|5|, |-2|\} = 2.$

(a) Butterworth filter

The order of normalized Butterworth filter is



$$N = \frac{\log\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\frac{\Omega_s}{\Omega_p}}$$

Given

$\alpha_p = 3$ dB, $\alpha_s = 20$ dB

$$\frac{\Omega_s}{\Omega_p} = \Omega_r = 2$$

$$= \frac{\log\sqrt{\frac{10^2-1}{10^{0.3}-1}}}{\log 2} = \frac{0.9978}{0.3010} = 3.32$$

Take $N = 4$.

For $N = 4$ the transfer function of Butterworth filter is

$$H(s) = \frac{1}{(s^2 + 0.765s + 1)(s^2 + 1.848s + 1)}$$

To get the transfer function of bandstop filter, use the transformation

$$s \to \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u \Omega_l}$$

$$i.e., s \to \frac{s(1000\pi)}{s^2 + 24 \times 10^4 \pi^2}$$

$$H(s)\Big|_{s \to \frac{1000\pi s}{s^2+24\times10^4\pi^2}}$$

$$= \frac{(s^2 + 24 \times 10^4\pi^2)^4}{[(1000\pi)^2 s^2 + (0.765 \times 1000\pi s)(s^2 + 24 \times 10^4\pi^2) + (s^2 + 24 \times 10^4\pi^2)^2] \; [(1000\pi)^2 s^2 + 1.848 \times 1000\pi s(s^2 + 24 \times 10^4\pi^2) + (s^2 + 24 \times 10^4\pi^2)]}$$

$$= \frac{(s + 24 \times 10^4\pi^2)^4}{[s^4 + 2403.32 s^3 + 1.4607 \times 10^7 s^2 + 5.69 \times 10^9 s + 5.61 \times 10^{12}] \; [s^4 + 5.805 \times 10^3 s^3 + 1.4607 \times 10^7 s^2 + 1.375 \times 10^{10} s + 5.61 \times 10^{12}]}$$

(b) Chebyshev filter

The order of the Chebyshev filter

$$N = \frac{\cos h^{-1}\sqrt{\frac{10^2-1}{10^{0.3}-1}}}{\cos h^{-1} 2}$$

$$= 2.75$$

Take $N = 3$

$$\varepsilon = (10^{0.1\alpha_p} - 1)^{0.5} = (10^{0.11} - 1)^{0.5} = 0.5368$$

$$\lambda = (10^{0.1\alpha_s} - 1)^{0.5} = (10^2 - 1)^{0.5} = 9.95$$

$$\mu = \varepsilon^{-1} + \sqrt{1 + \varepsilon^{-2}} = 3.97$$

$$a = \Omega_p \left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right] \quad ; \quad b = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right]$$

$$= 1\left[\frac{(3.97)^{1/3} - (3.97)^{-1/3}}{2}\right] \qquad = 1\left[\frac{(3.97)^{1/3} + (3.97)^{-1/3}}{2}\right]$$

$$= 0.476 \qquad\qquad = 1.107$$

($\Omega_p = 1$for normalized Chebyshev filter)

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \qquad k = 1, 2, 3.$$

$$\phi_1 = \frac{\pi}{2} + \frac{\pi}{6} = 120°$$

$$\phi_2 = \frac{\pi}{2} + \frac{\pi}{2} = 180°$$

$$\phi_3 = \frac{\pi}{2} + \frac{5\pi}{6} = 240°$$

$$s_1 = a\cos\phi_1 + jb\sin\phi_1$$
$$= 0.476\cos 120° + j1.107\sin 120°$$
$$= -0.238 + j0.9586$$
$$s_2 = a\cos\phi_2 + jb\sin\phi_2$$
$$= (0.476)\cos 180° + j1.107\sin 180°$$
$$= -0.476$$
$$s_3 = a\cos\phi_3 + jb\sin\phi_3$$
$$= 0.476\cos 240° + j1.107\sin 240°$$
$$= -0.238 - j0.9586$$

Denominator of the transfer function

$$= (s + 0.476)\{(s + 0.238)^2 + (0.9586)^2\}$$
$$= (s + 0.476)(s^2 + 0.476s + 0.975)$$

Numerator of the transfer function

$$= (0.476)(0.9755) = 0.46463$$
$$H(s) = \frac{0.4643}{(s + 0.476)(s^2 + 0.476s + 0.9755)}$$

The transfer function of Bandstop filter can be obtained by using the following ansformation

$$H(s)\Big|_{s \to \frac{s(1000\pi)}{s^2 + 24\times 10^4\pi^2}}$$

$$= \frac{0.4643(s^2 + 24\times 10^4\pi^2)^3}{[1000\pi s + 0.476(s^2 + 24\times 10^4\pi^2)]\,[s^2(1000\pi)^2 + 0.476(1000\pi s)(s^2 + 24\times 10^4\pi^2) + 0.9755(s^2 + 24\times 10^4\pi^2)^2]}$$

$$= \frac{0.4643(s^2 + 24\times 10^4\pi^2)^3}{(0.476s^2 + 1000\pi s + 1.1275\times 10^6)(0.9755s^4 + 4.621\times 10^6 + 5.47\times 10^{12} + 9.369\times 10^6\pi^2 s^2 + 1495.4s^3 + 3.5\times 10^9 s)}$$

$$= \frac{(s^2 + 24\times 10^4\pi^2)^3}{(s^2 + 6600s + 2.3687\times 10^6)(s^4 + 1533s^3 + 1.45\times 10^7 s^2 + 3.589\times 10^9 s + 5.6\times 10^{12})}$$

Example 5.36 Using bilinear transformation design a digital bandpass Butterworth filter with the following specifications

Sampling frequency F = 8 KHz

$\alpha_p = 2$ dB in the passband 800Hz $\leq f \leq$ 1000 Hz

$\alpha_s = 20$ dB in the stopband $0 \leq f \leq$ 400 Hz and 2000 Hz $\leq f \leq \infty$

Solution

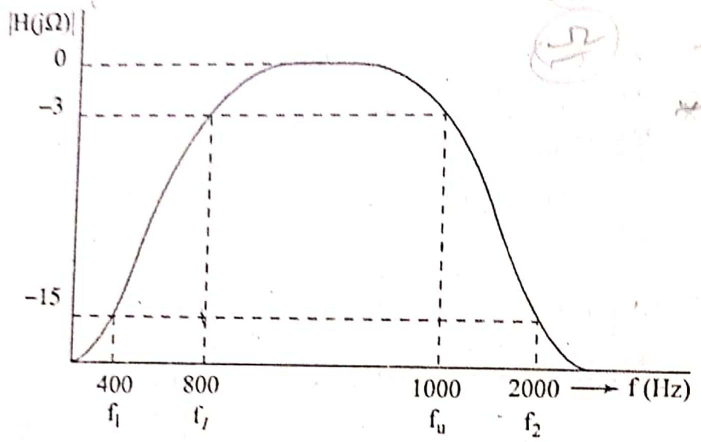


Fig. 5.65

$$\frac{\omega_1 T}{2} = \frac{2\times\pi\times 400}{2\times 8000} = \frac{\pi}{20}$$
$$\frac{\omega_l T}{2} = \frac{2\pi\times 800}{2\times 8000} = \frac{\pi}{10}$$
$$\frac{\omega_u T}{2} = \frac{2\times\pi\times 1600}{2\times 8000} = \frac{\pi}{5}$$
$$\frac{\omega_2 T}{2} = \frac{2\times\pi\times 2000}{2\times 8000} = \frac{\pi}{4}$$

Prewarped analog frequencies are given by

$$\frac{\Omega_1 T}{2} = \tan\frac{\omega_1 T}{2} = \tan\frac{\pi}{20} = 0.1584$$
$$\frac{\Omega_l T}{2} = \tan\frac{\omega_l T}{2} = \tan\frac{\pi}{10} = 0.325$$
$$\frac{\Omega_u T}{2} = \tan\frac{\omega_u T}{2} = \tan\frac{\pi}{5} = 0.7265$$
$$\frac{\Omega_2 T}{2} = \tan\frac{\omega_2 T}{2} = \tan\frac{\pi}{4} = 1$$

First we design a prototype normalized lowpass filter and then use suitable transformation to obtain the transfer function of bandpass filter.

To reduce computational complexity we use above values to find Ω_r and substitute $s = \frac{1-z^{-1}}{1+z^{-1}}$ for bilinear transformation ($\because$ all the above frequencies contains the term $\frac{T}{2}$).

We have

$$A = \frac{-\Omega_1^2 + \Omega_l\Omega_u}{\Omega_1(\Omega_u - \Omega_l)};$$
$$= \frac{-(0.1584)^2 + (0.325)(0.7265)}{0.1584(0.7265 - 0.325)}$$
$$= 3.318$$

$$B = \frac{\Omega_2^2 - \Omega_l\Omega_u}{\Omega_2(\Omega_u - \Omega_l)}.$$
$$= \frac{1 - (0.7265)(0.325)}{1(0.7265 - 0.325)}$$
$$= 1.90258$$

$\Omega_r = \min\{|A|, |B|\} = 1.90258$

$$N = \frac{\log_{10}\sqrt{\frac{10^2-1}{10^{0.2}-1}}}{\log_{10}(1.90258)} = 3.9889$$

Let us choose $N = 4$.

The Fourth order normalized Butterworth lowpass filter transfer function is given by

$$H(s) = \frac{1}{(s^2 + 0.76537s + 1)(s^2 + 1.84776s + 1)}$$

The transformation for the bandpass filter is

$$s \rightarrow \frac{s^2 + \Omega_l\Omega_u}{s(\Omega_u - \Omega_l)} = \frac{s^2 + 0.236}{s(0.402)}$$

$$H(s) = \left.\frac{1}{(s^2 + 0.76537s + 1)(s^2 + 1.84776s + 1)}\right|_{s \rightarrow \frac{s^2+0.236}{s(0.402)}}$$

$$= \frac{1}{\left[\left(\frac{s^2+0.236}{0.402s}\right)^2 + 0.76537\left(\frac{s^2+0.236}{0.402s}\right) + 1\right]\left[\left(\frac{s+0.236}{0.402s}\right)^2 + 1.84776\left(\frac{s^2+0.236}{0.402s}\right) + 1\right]}$$

$$= \frac{0.0261s^4}{(s^4 + 0.30768s^3 + 0.6336s^2 + 0.0726s + 0.055696)(s^4 + 0.7428s^3 + 0.6336s^2 + 0.1753s + 0.055696)}$$

$$H(z) = H(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}}$$

$$H(z) = \frac{0.0261(1-z^{-1})^4}{5.3962 - 21.2398z^{-1} + 44.566z^{-2} - 60.512z^{-3} + 58.1635z^{-4} - 39.86z^{-5} + 19.28z^{-6} - 6.0087z^{-7} + 1.009z^{-8}}$$

$$= \frac{0.004837(1-z^{-1})^4}{1 - 3.936z^{-1} + 8.2587z^{-2} - 11.214z^{-3} + 10.778z^{-4} - 7.3866z^{-5} + 3.573z^{-6} - 1.1135z^{-7} + 0.187z^{-8}}$$

0.002 + 0.001

0.000002

Example 5.40 The normalized transfer function of an analog filter is given by

$$H_a(s_n) = \frac{1}{s_n^2 + 1.414\, s_n + 1}$$

Convert the analog filter to a digital with a cut off frequency of 0.4π using bilinear transformation

AU 2006 CSE

Solution

$$\Omega_c = \frac{2}{T}\tan\frac{\omega_c}{2} = 2\tan(0.2\pi) = 1.45$$

$$H_a(s) = \frac{1}{\left(\frac{s}{1.45}\right)^2 + 1.414\left(\frac{s}{1.45}\right) + 1} = \frac{(1.45)^2}{s^2 + 2.055s + (1.45)^2}$$

$$H(z) = H_a(s)\quad s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$H(z) = \frac{2.1}{\frac{4(1-z^{-1})^2}{(1+z^{-1})^2} + (2.055)\,2\frac{(1-z^{-1})}{1+z^{-1}} + 2.1}$$

$$= \frac{2.1(1+z^{-1})^2}{4(1-z^{-1})^2 + 4.11(1-z^{-2}) + 2.1(1+z^{-1})^2}$$

$$= \frac{2.1(1+z^{-1})^2}{4(1-2z^{-1}+z^{-2}) + 4.11(1-z^{-2}) + 2.1(1+2z^{-1}+z^{-2})}$$

$$= \frac{2.1(1+z^{-1})^2}{10.21 - 3.8z^{-1} + 1.99z^{-2}}$$

$$= \frac{0.2(1+z^{-1})^2}{1 - 0.37z^{-1} + 0.195z^{-2}}$$

Example 5.41 **(i) Enumerate the various steps involved in the design of low pass digital Butterworth IIR filter. (ii) The specification of the desired low pass filter is**

$$0.8 \le \left|H(e^{j\omega})\right| \le 1.0;\ 0 \le \omega \le 0.2\pi$$
$$\left|H(e^{j\omega})\right| \le 0.2;\ \ 0.32\pi \le \omega \le \pi$$

Design Butterworth digital filter using impulse invariant transformation

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Solution

$$\frac{1}{\sqrt{1+\varepsilon^2}} = 0.8 \quad\Rightarrow\quad \varepsilon = 0.75$$

$$\frac{1}{\sqrt{1+\lambda^2}} = 0.2 \quad\Rightarrow\quad \lambda = 4.898$$

$$N = \frac{\log \frac{\lambda}{\varepsilon}}{\log \frac{\Omega_s}{\Omega_{j0}}} = 3.99$$

Select N = 4

$$H(s) = \frac{1}{(s^2 + 0.76537s + 1)(s^2 + 1.8477s + 1)}$$

$$\Omega_c = \frac{\Omega_s}{\lambda^{1/N}} = \frac{0.32\pi}{(4.898)^{1/N}} = 0.2151\pi = 0.6757$$

$$H_a(s) = \frac{0.2084}{(s^2 + 0.5171s + 0.4565)(s^2 + 1.2485s + 0.4565)}$$

The poles are $-0.2585 \pm j0.6242$ and $0.6242 \pm j\,0.2585$

$$H_a(s) = \frac{A}{s + 0.2585 - j0.6242} + \frac{A^*}{s + 0.2585 + j0.6242} + \frac{B}{s + 0.6242 - j0.2585} + \frac{B^*}{s + 0.6242 + j0.2585}$$

Solving for $A = -0.3121 + j0.129$

$B = 0.3121 - j0.7538$

$$H(z) = \frac{-0.3121 + j0.129}{1 - e^{-0.2585} e^{j0.6242} z^{-1}} + \frac{(-0.3121 - j0.129)}{1 - e^{-0.2585} e^{-j0.6242} z^{-1}} + \frac{0.3121 - j0.7538}{1 - e^{-0.6242} e^{j0.2585} z^{-1}} + \frac{0.3121 + j0.7538}{1 - e^{-0.6242} e^{-j0.2585} z^{-1}}$$

$$= \frac{-0.6242 + 0.2747\, z^{-1}}{1 - 0.253\, z^{-1} + 0.5963\, z^{-2}} + \frac{0.6242 - 0.1168\, z^{-1}}{1 - 1.03582 \cdot z^{-1} + 0.2869\, z^{-2}}$$